Extended Hermite Subdivision Schemes

Jean-Louis Merrien, Tomas Sauer

Extended Hermite Subdivision Schemes

Jean-Louis Merrien[*], Tomas Sauer[†]

September 1, 2016

Abstract

Subdivision schemes are efficient tools for building curves and surfaces. For vector subdivision schemes, it is not so straightforward to prove more than the Hölder regularity of the limit function. On the other hand, Hermite subdivision schemes produce function vectors that consist of derivatives of a certain function, so that the notion of convergence automatically includes regularity of the limit. In this paper, we establish an equivalence between a spectral condition and operator factorizations, then we study how such schemes with smooth limit functions can be extended into ones with higher regularity. We conclude by pointing out this new approach applied to cardinal splines.

keywords: Subdivision, Hermite, Convergence, Derivatives.

1 Introduction and Notations

In recent decades, subdivision schemes [1, 11] have resulted in efficient methods to generate smooth curves and surfaces, for example, for the purpose of computer aided design.

In *vector subdivision schemes* [21], one starts with a sequence of vector valued data at level 0, and iteratively applies a stationary subdivision rule to generate a new sets of vectors on the finer and finer grids $2^{-n}\mathbb{Z}$. In this process, the data $\boldsymbol{f}_n$ is interpreted as an approximation of a limit function $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^p$ on the grid where p is a positive integer, i.e.,

$$\boldsymbol{f}_n(\alpha) \simeq \boldsymbol{f}\left(2^{-n}\alpha\right), \qquad \alpha \in \mathbb{Z}.$$

[*]INSA de Rennes, 20 avenue des Buttes de Coesmes, CS 14315, 35043 RENNES CEDEX, France, email: jmerrien@insa-rennes.fr

[†]Lehrstuhl für Mathematik mit Schwerpunkt Digitale Bildverarbeitung & FORWISS, Universität Passau, Innstr. 43, D-94032 PASSAU, email: Tomas.Sauer@fim.uni-passau.de

Under suitable conditions on the subdivision rule, this sequence of vectors then converges to a limit curve or surface, either uniformly or in some L_p norm. In this context, the results are vector valued functions with some Hölder regularity related to the convergence rate of the process, but essentially they are "simply" continuous or L_p functions. The smoothness of the limit functions in the sense of differentiability implies additional conditions for the subdivision scheme which cannot be easily verified, cf. [2, 3, 4, 21].

Hermite subdivision schemes where the components of the vector valued function are the successive exact or approximated derivatives of the first component, approach subdivision from a different point of view cf. [9, 10, 13, 14, 15, 16]. Besides the fact that now the subdivision operator varies with the level of iteration, the regularity of the limit function is part of the definition of convergence and, therefore, the verification of smoothness properties is already included in the convergence proof.

Convergence of subdivision schemes is usually described by identifying certain special eigenvectors of the subdivision operator which imply a factorization of the subdivision operator by means of an operator that annihilates this invariant space. In the scalar case, this annihilator is simply a difference operator, in the vector case, it is a more intricate difference based on the *rank* of the subdivision scheme [21] and in Hermite subdivision it is the *Taylor operator*, cf. [8, 18]. All of these operators are in a one-to-one relationship with the sum rules or the spectral conditions for the subdivision operator.

In this paper, we study these relationships for Hermite schemes. We show that spectral condition is equivalent to factorization, once a certain normalization property is satisfied; in a previous paper we had only proved that spectral condition implies factorization. After that, we explore how the regularity of the limit can be increased by an *extended scheme* that transfers a given Hermite scheme to one of higher degree. After a first example, the extension process and the factorization are studied in detail.

As an example, we propose a new approach to cardinal splines [22]. Starting with transformations to a scalar subdivision scheme [20], we describe extended Hermite schemes of different degrees. We give new eigenvalue and eigenpolynomial properties and we prove that the maximum regularity of the spline can be recovered by this new process.

The paper is organized as follows: in Section 2, we recall the definition of a Hermite subdivision scheme and we define the key to transform this scheme into a vector subdivision scheme, namely the spectral condition. Having shown in earlier work [18] that the spectral condition implies an important and useful factorization in terms of *Taylor operators*, we now even prove, in Corollary 2.12, a characterization of the spectral condition in terms of factorization and a normalization property which will become useful for studying extended schemes. In Section 3, we introduce extended schemes and consider some of their general properties; the goal of extensions is to exploit higher order regularity of limit functions by adding further components and therefore further derivatives to the scheme. In Sections 4 and 5, two particular extensions are studied in more detail, one from de Rham scheme [6, 7], one from cardinal splines. These examples demonstrate the potential of the extension approach and thus motivate and justify this construction.

As for notation, vectors in $\mathbb{R}^r$ will be labeled by lowercase boldface letters: $\boldsymbol{y} = [y_j]_{j=0,\dots,r-1}$ or $\boldsymbol{y} = \left[y^{(j)}\right]_{j=0,\dots,r-1}$, where we use the latter notation to highlight the fact that in Hermite subdivision the components of the vectors correspond to derivatives. Matrices in $\mathbb{R}^{r\times r}$ will be written as uppercase boldface letters, such as $\boldsymbol{A} = [a_{jk}]_{j,k=0,\dots,r-1}$. The space of polynomials in one variable of degree at most n will be written as $\mathcal{P}_n$. Vector sequences will be considered as functions from $\mathbb{Z}$ to $\mathbb{R}^r$ and the vector space of all such functions will be denoted by $\ell(\mathbb{Z},\mathbb{R}^r)$ or $\ell^r(\mathbb{Z})$. For $\boldsymbol{y}(\cdot) \in \ell(\mathbb{Z},\mathbb{R}^r)$, the *forward difference* is defined as $\Delta\boldsymbol{y}(\alpha) := \boldsymbol{y}(\alpha+1) - \boldsymbol{y}(\alpha)$, $\alpha \in \mathbb{Z}$, and iterated to $\Delta^{i+1}\boldsymbol{y} := \Delta\left(\Delta^i\boldsymbol{y}\right) = \Delta^i\boldsymbol{y}(\cdot+1) - \Delta^i\boldsymbol{y}(\cdot)$, $i \geq 0$.

By $\boldsymbol{\mathcal{A}} \in \ell^{r\times r}(\mathbb{Z})$, we denote a sequence of matrices, that is, for $\alpha \in \mathbb{Z}$ the sequence element $\boldsymbol{A}(\alpha) = [a_{jk}(\alpha)]_{j,k=0,\dots,r-1}$ is an $r\times r$ matrix. Any such sequence will be called a *mask* provided that it is finitely supported, that is, there exists $N \in \mathbb{N}$ such that

$$\operatorname{supp}\boldsymbol{\mathcal{A}} := \{\alpha \in \mathbb{Z} \ : \ \boldsymbol{A}(\alpha) \neq 0\} \subseteq [-N,N].$$

To any mask $\boldsymbol{\mathcal{A}}$, we associate the *stationary vector subdivision operator* $S_{\boldsymbol{\mathcal{A}}} : \ell^r(\mathbb{Z}) \to \ell^r(\mathbb{Z})$, defined as

$$S_{\boldsymbol{\mathcal{A}}}\boldsymbol{c}(\alpha) := \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\alpha-2\beta)\,\boldsymbol{c}(\beta), \qquad \alpha \in \mathbb{Z}, \qquad \boldsymbol{c} \in \ell^r(\mathbb{Z}), \tag{1}$$

and also its *symbol* which is the Laurent polynomial

$$\boldsymbol{\mathcal{A}}^*(z) := \sum_{\alpha\in\mathbb{Z}} \boldsymbol{A}(\alpha) z^\alpha, \qquad z \in \mathbb{C}\setminus\{0\}. \tag{2}$$

We also recall the *shift relation*

$$\big(S_{\boldsymbol{\mathcal{A}}}\boldsymbol{c}\big)(\cdot+2) = S_{\boldsymbol{\mathcal{A}}}\boldsymbol{c}(\cdot+1). \tag{3}$$

The *stationary subdivision scheme* associated with the subdivision operator $S_{\boldsymbol{\mathcal{A}}}$ is given by the repeated application of the operator, starting with any initial sequence $\boldsymbol{c}_0 \in \ell_\infty^r(\mathbb{Z})$, which stands for the space of all uniformly bounded r–vector valued biinfinite sequences,

$$\boldsymbol{c}_{n+1} := S_{\boldsymbol{\mathcal{A}}} \boldsymbol{c}_n \,, \qquad n \geq 0. \tag{4}$$

If we let the masks vary with the iteration level n, which is sometimes called *non–stationary subdivision scheme*, we get a level dependent scheme of the form

$$\boldsymbol{c}_{n+1}(\alpha) := \sum_{\beta \in \mathbb{Z}} \boldsymbol{A}_n \left(\alpha - 2\beta\right) \, \boldsymbol{c}_n(\beta), \qquad \alpha \in \mathbb{Z}, \qquad \boldsymbol{c} \in \ell^r \left(\mathbb{Z}\right), \tag{5}$$

where $\boldsymbol{\mathcal{A}}_n$ is a matrix valued biinfinite sequence for $n \in \mathbb{N}$.

2 Hermite subdivision schemes

A special type of a level dependent vector subdivision scheme called *Hermite subdivision scheme* $H_{\boldsymbol{\mathcal{A}}}$ of degree d is given by the following construction: Starting with $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$, for $n \in \mathbb{N}$, we define $\boldsymbol{f}_{n+1} \in \ell^{d+1}(\mathbb{Z})$ by

$$\boldsymbol{D}^{n+1} \boldsymbol{f}_{n+1}(\alpha) = S_{\boldsymbol{\mathcal{A}}} \boldsymbol{D}^n \boldsymbol{f}_n(\alpha) = \sum_{\beta \in \mathbb{Z}} \boldsymbol{A} \left(\alpha - 2\beta\right) \, \boldsymbol{D}^n \, \boldsymbol{f}_n(\beta), \qquad \alpha \in \mathbb{Z}, \tag{6}$$

where

$$\boldsymbol{D} := \begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2^d} \end{bmatrix}$$

is the diagonal matrix with diagonal entries 2^{-j}, respectively, $j = 0, \ldots, d$, that represents the influence of scaling on the successive derivatives.

Remark 2.1 *It is important to observe that (6) can also be written as*

$$\boldsymbol{f}_{n+1}(\alpha) = S_{\boldsymbol{\mathcal{A}}_n} \boldsymbol{f}_n(\alpha) = \sum_{\beta \in \mathbb{Z}} \boldsymbol{D}^{-(n+1)} \boldsymbol{A} \left(\alpha - 2\beta\right) \, \boldsymbol{D}^n \, \boldsymbol{f}_n(\beta), \tag{7}$$

which involves the level dependent masks $\boldsymbol{\mathcal{A}}_n := \{\boldsymbol{D}^{-(n+1)} \boldsymbol{A}(\alpha) \boldsymbol{D}^n\}_{\alpha \in \mathbb{Z}}$, $n \in \mathbb{N}$.

Remark 2.2 *We notice that if $\varphi \in C^d(\mathbb{R})$ and $\varphi_n(x) := \varphi(x/2^n)$, then*

$$\begin{bmatrix} \varphi_n(x) \\ \varphi_n'(x) \\ \vdots \\ \varphi_n^{(d)}(x) \end{bmatrix} = \boldsymbol{D}^n \begin{bmatrix} \varphi(x/2^n) \\ \varphi'(x/2^n) \\ \vdots \\ \varphi^{(d)}(x/2^n) \end{bmatrix}.$$

For the iterated application of a Hermite subdivision scheme, the first component $f_n^{(0)}(\beta)$ can be interpreted as the approximation of $\varphi(\beta/2^n)$, while the next one, $f_n^{(1)}(\beta)$, describes the approximation of the derivative $\varphi'(\beta/2^n)$, and so on, up to the last one, $f_n^{(d)}(\beta)$ which is an approximation of $\varphi^{(d)}(\beta/2^n)$.

Finally, $H_{\boldsymbol{\mathcal{A}}}$ will be called *interpolatory* if at each step we have $\boldsymbol{f}_{n+1}(2\cdot) = \boldsymbol{f}_n(\cdot)$, which is equivalent to $\boldsymbol{A}(2\beta) = \boldsymbol{D}\,\delta_{\beta,0}$ for $\beta \in \mathbb{Z}$.

2.1 Spectral condition

To any function $\varphi \in C^d(\mathbb{R})$, we associate the vector sequence $\boldsymbol{v}_\varphi \in \ell^{d+1}(\mathbb{Z})$ of the samples of the function and its derivatives up to order d on $\mathbb{Z}$, defined by

$$\boldsymbol{v}_\varphi(\alpha) := \begin{bmatrix} \varphi(\alpha) \\ \varphi'(\alpha) \\ \vdots \\ \varphi^{(d)}(\alpha) \end{bmatrix}, \qquad \alpha \in \mathbb{Z}. \tag{8}$$

$H_{\boldsymbol{\mathcal{A}}}$ *reproduces* a function $\varphi \in C^d(\mathbb{R})$ if for the initial value $\boldsymbol{f}_0 = \boldsymbol{v}_\varphi$ we obtain $\boldsymbol{f}_n(\cdot) = \boldsymbol{v}_\varphi(\cdot/2^n)$ for any $n \in \mathbb{N}$. At this point, it is already worthwhile to recall that a C^d-convergent interpolatory scheme of degree d reproduces any polynomial of $\mathcal{P}_d$, see [10]; the will be given later in Definition 2.13.

A fundamental property of Hermite subdivision schemes is the *spectral condition*, introduced in [8], which requires the existence of particular *polynomial* eigenvalues of the stationary subdivision operator $S_{\boldsymbol{\mathcal{A}}}$ that affect the behavior of the Hermite scheme in a crucial way.

Definition 2.3 *A mask $\boldsymbol{\mathcal{A}}$ or its associated subdivision operator $S_{\boldsymbol{\mathcal{A}}}$ satisfies the* spectral condition *of order ℓ if for $j = 0, \dots, \ell$, there exist polynomials p_j of exact degree j such that*

$$S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_j} = \frac{1}{2^j} \boldsymbol{v}_{p_j}. \tag{9}$$

If (9) *holds true, we will always assume that p_j is normalized such that $p_j(x) = \frac{1}{j!}x^j + q_j(x)$ with $q_j \in \mathcal{P}_{j-1}$ for $j > 0$ and $q_0 = 0$.*

Remark 2.4 *Though the spectral condition is a property of the stationary subdivision scheme, $S_{\boldsymbol{\mathcal{A}}}$, it is still linked with the derivatives of polynomials. Moreover, it was proved in [8] that the spectral condition is also equivalent to the* sum rule *introduced by Bin Han and his collaborators in [14, 15].*

The following result is easily obtained by using linearity to ensure the reproduction of any basis of polynomials and especially the $p_j(x) := x^j/j!$ for $j = 0, \dots, \ell$.

Proposition 2.5 *If $H_{\boldsymbol{\mathcal{A}}}$ reproduces a basis of $\mathcal{P}_\ell$ then $S_{\boldsymbol{\mathcal{A}}}$ satisfies the spectral condition of order ℓ for the polynomials $p_j(x) := \dfrac{x^j}{j!}$, $j = 0, \dots, \ell$.*

Definition 2.6 *The* Taylor operator T_d *and the* complete Taylor operator $\widetilde{T}_d$ *of degree d, mapping $\ell^{(d+1)}(\mathbb{Z})$ to itself are defined as*

$$T_d := \begin{bmatrix} \Delta & -1 & \dots & -\frac{1}{(d-1)!} & -\frac{1}{d!} \\ & \Delta & \ddots & \vdots & \vdots \\ & & \ddots & -1 & \vdots \\ & & & \Delta & -1 \\ & & & & 1 \end{bmatrix}, \quad \widetilde{T}_d := \begin{bmatrix} \Delta & -1 & \dots & -\frac{1}{(d-1)!} & -\frac{1}{d!} \\ & \Delta & \ddots & \vdots & \vdots \\ & & \ddots & -1 & \vdots \\ & & & \Delta & -1 \\ & & & & \Delta \end{bmatrix}.$$

where the constants in the matrices are to be understood as multiples of the identity.

Proposition 2.7 *For any $p \in \mathcal{P}_d$, we have $T_d \boldsymbol{v}_p = [0, \dots, 0, p^{(d)}]^T \in \mathbb{R}^{d+1}$ and $\widetilde{T}_d \boldsymbol{v}_p = \mathbf{0} \in \mathbb{R}^{d+1}$.*

Proof: If $p \in \mathcal{P}_d$, then its $d+1$-th derivative is 0. Thus for $j = 0, \dots, d-1$, the Taylor expansion of $p^{(j)}(\alpha + 1)$ at point α is $p^{(j)}(\alpha + 1) = \sum_{i=0}^{d-j} \frac{1}{i!} p^{(j+i)}(\alpha)$, which gives that the j-th row of $T_d \boldsymbol{v}_p(\alpha)$ or $\widetilde{T}_d \boldsymbol{v}_p$ is 0. The last row of $T_d \boldsymbol{v}_p(\alpha)$ or $\widetilde{T}_d \boldsymbol{v}_p(\alpha))$ is $p^{(d)}(\alpha)$ or $p^{(d)}(\alpha + 1) - p^{(d)}(\alpha) = 0$, respectively. □

The following factorization result has been proved, among others, in [18]. In particular, the explicit construction of the "factor masks" $\boldsymbol{\mathcal{B}}$ and $\widetilde{\boldsymbol{\mathcal{B}}}$ and its algebraic background have been pointed out there. Here, we slightly extend the result by adding a normalization property that will eventually enable us to also give a converse statement.

Theorem 2.8 *If the mask $\boldsymbol{\mathcal{A}} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ satisfies the spectral condition of order at least d, then there exist two finitely supported masks $\boldsymbol{\mathcal{B}}$ and $\widetilde{\boldsymbol{\mathcal{B}}}$ in $\ell^{(d+1)\times(d+1)}(\mathbb{Z})$ such that*

$$T_d S_{\boldsymbol{\mathcal{A}}} = 2^{-d} S_{\boldsymbol{\mathcal{B}}} T_d, \qquad \textit{and} \qquad \widetilde{T}_d S_{\boldsymbol{\mathcal{A}}} = 2^{-d} S_{\widetilde{\boldsymbol{\mathcal{B}}}} \widetilde{T}_d. \tag{10}$$

Moreover,

$$S_{\boldsymbol{\mathcal{B}}} \boldsymbol{e}_d = \boldsymbol{e}_d := [0, \ldots, 0, 1]^T \in \mathbb{R}^{d+1}. \tag{11}$$

Proof: The only thing to be proved is the normalization, i.e. $S_{\boldsymbol{\mathcal{B}}} \boldsymbol{e}_d = \boldsymbol{e}_d$. Let $p_d \in \mathcal{P}_d$ be the normalized polynomial such that $\frac{1}{2^d} \boldsymbol{v}_{p_d} = S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_d}$. In Proposition 2.7, we have seen that $T_d \boldsymbol{v}_{p_d} = [0, \ldots, 0, 1]^T = \boldsymbol{e}_d$. The normalization follows from $\boldsymbol{e}_d = T_d \boldsymbol{v}_{p_d} = 2^d T_d S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_d} = S_{\boldsymbol{\mathcal{B}}} T_d \boldsymbol{v}_{p_d} = S_{\boldsymbol{\mathcal{B}}} \boldsymbol{e}_d$. $\square$

Next, we will give a converse to Theorem 2.8, stated and proved in Theorem 2.11, that shows that the incomplete Taylor factorization does in fact characterize the spectral condition of order d as long as the additional normalization condition (11) from Theorem 2.8 is satisfied. A few preliminary results are required.

Proposition 2.9 *A sequence $\boldsymbol{v} \in \ell^{d+1}(\mathbb{Z})$ satisfies $\widetilde{T}_d \boldsymbol{v} = 0$ if and only if $\boldsymbol{v} = \boldsymbol{v}_p$ for some $p \in \mathcal{P}_d$.*

Proof: Since the direction "$\Leftarrow$" is given in Proposition 2.7, we only need to prove "$\Rightarrow$" for which we use an idea from [8, Lemma 4]. Let $\boldsymbol{v} \in \ell^{d+1}(\mathbb{Z})$ such that $\widetilde{T}_d \boldsymbol{v} = 0$. For a given $\alpha \in \mathbb{Z}$, we define

$$p(x) = \sum_{k=0}^{d} \frac{v_k(\alpha)}{k!} (x - \alpha)^k$$

and the corresponding sampling vector $\boldsymbol{v}_p$. Since for $k = 0, \ldots, d$, $p^{(k)}(\alpha) = v_k(\alpha)$, we notice that $\boldsymbol{v}_p(\alpha) - \boldsymbol{v}(\alpha) = \mathbf{0}$.

The hypothesis gives $\widetilde{T}_d \boldsymbol{v} = \mathbf{0}$ and, by Proposition 2.7, also $\widetilde{T}_d \boldsymbol{v}_p = \mathbf{0}$. Thus, for $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_p$ we obtain that $\widetilde{T}_d \boldsymbol{u} = \mathbf{0}$ or

$$u_k(\cdot + 1) = \sum_{j=k}^{d} \frac{u_k(\cdot)}{(j-k)!}, \qquad k = 0, \ldots, d.$$

Then an inductive reasoning, starting from $\boldsymbol{u}(\alpha) = \mathbf{0}$, yields that $\boldsymbol{u}(\beta) = \mathbf{0}$ or $\boldsymbol{v}(\beta) = \boldsymbol{v}_p(\beta)$ for any integer $\beta \geq \alpha$.

If we choose α' instead of α and corresponding polynomial q instead of p, we obtain for any $\beta \geq \max(\alpha, \alpha')$ that $\boldsymbol{v}(\beta) = \boldsymbol{v}_p(\beta) = \boldsymbol{v}_q(\beta)$, hence $p(\beta) = q(\beta)$ for any sufficiently large β and therefore, for all $\beta \in \mathbb{Z}$. Consequently, $p = q$ and especially $\boldsymbol{v}(\alpha') = \boldsymbol{v}_p(\alpha') = \boldsymbol{v}_q(\alpha')$. Now we fix $\alpha = 0$. Then $\boldsymbol{v}(\beta) = \boldsymbol{v}_p(\beta)$ for any $\beta \geq 0$. If $\beta < 0$, we fix $\alpha' = \beta$ and we have seen that $\boldsymbol{v}(\alpha') = \boldsymbol{v}_p(\alpha')$. Hence, we can conclude that $\boldsymbol{v} = \boldsymbol{v}_p$. $\square$

Lemma 2.10 *Let $p_\ell := (\cdot)^\ell/\ell!$, $0 \leq \ell \leq d$. Then*

$$\Delta_\beta \boldsymbol{v}_{p_\ell} := \boldsymbol{v}_{p_\ell}(\cdot + \beta) - \boldsymbol{v}_{p_\ell} = \sum_{k=0}^{\ell-1} \frac{\beta^{\ell-k}}{(\ell-k)!} \boldsymbol{v}_{p_k}. \tag{12}$$

Proof: For $j \leq \ell$ we have

$$\frac{(x+\beta)^j}{j!} - \frac{x^j}{j!} = \frac{1}{j!} \sum_{k=0}^{j-1} \binom{j}{k} \beta^{j-k} x^k = \sum_{k=0}^{j-1} \frac{\beta^{j-k}}{(j-k)!} \frac{x^k}{k!} =: q_j^*(x),$$

hence, with $q_j^* = 0$, $j = \ell+1, \ldots, d$,

$$\boldsymbol{v}_{p_\ell}(\cdot + \beta) - \boldsymbol{v}_{p_\ell} = \begin{bmatrix} q_0^* \\ \vdots \\ q_d^* \end{bmatrix}.$$

On the other hand, for $r \leq \ell - j - 1$,

$$\frac{d^r}{dx^r} q_j^*(x) = \sum_{k=r}^{j-1} \frac{\beta^{j-k} x^{k-r}}{(k-r)!(j-k)!} = \sum_{k=0}^{j-1-r} \frac{\beta^{j-k-r}}{(j-k-r)!} \frac{x^k}{k!} = q_{j+r}^*(x),$$

and setting $q_{\beta,\ell} := q_\ell^*$ yields

$$q_{\beta,\ell} = \sum_{k=0}^{\ell-1} \frac{\beta^{\ell-k}}{(\ell-k)!} \frac{(\cdot)^k}{k!} = \sum_{k=0}^{\ell-1} \frac{\beta^{\ell-k}}{(\ell-k)!} p_k$$

and proves the claim. $\square$

Theorem 2.11 *Let $\boldsymbol{\mathcal{A}} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$. If*

1. $S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_p = \boldsymbol{v}_q$, $\deg q \leq \deg p$, *for any* $p \in \mathcal{P}_{d-1}$,
2. *There exists a finitely supported* $\boldsymbol{\mathcal{B}} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ *such that* (10) *holds true, i.e.,* $T_d S_{\boldsymbol{\mathcal{A}}} = 2^{-d} S_{\boldsymbol{\mathcal{B}}} T_d$,

3. $S_{\boldsymbol{\mathcal{B}}}\boldsymbol{e}_d = \boldsymbol{e}_d$,

then $\boldsymbol{\mathcal{A}}$ *satisfies the spectral condition of order* d.

Proof: Since

$$T_d \boldsymbol{v}_{p_d} = \boldsymbol{e}_d = S_{\boldsymbol{\mathcal{B}}} e_d = S_{\boldsymbol{\mathcal{B}}} T_d \boldsymbol{v}_{p_d} = 2^d T_d S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_d},$$

i.e., $T_d \left(S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_d} - 2^{-d} \boldsymbol{v}_{p_d} \right) = 0$, which yields that $S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_d} = 2^{-d} \boldsymbol{v}_{p_d} + \boldsymbol{v}'$ with

$$0 = T_d \boldsymbol{v}' = \begin{bmatrix} \widetilde{T}_{d-1} & * \\ 0 & 1 \end{bmatrix} \boldsymbol{v}'.$$

Hence, $v'_d = 0$ and $\widetilde{T}_{d-1} \boldsymbol{v}'_{0:d-1} = 0$, so that Proposition 2.9 implies that there exists $\widetilde{q}_{d-1} \in \mathcal{P}_{d-1}$ such that

$$S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_d} = 2^{-d} \boldsymbol{v}_{p_d} + \boldsymbol{v}_{\widetilde{q}_{d-1}}. \tag{13}$$

Since $S_{\boldsymbol{\mathcal{A}}}$ is a stationary subdivision operator with scaling factor 2, we have $S_{\boldsymbol{\mathcal{A}}} \boldsymbol{c}(\cdot + 2) = S_{\boldsymbol{\mathcal{A}}} \left(\boldsymbol{c}(\cdot + 1) \right)$, hence, by Lemma 2.10,

$$\Delta_2 S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_d} = S_{\boldsymbol{\mathcal{A}}} \left(\boldsymbol{v}_{p_d}(\cdot + 1) - \boldsymbol{v}_{p_d} \right) = S_{\boldsymbol{\mathcal{A}}} \Delta \boldsymbol{v}_{p_d} = S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_{d-1}} + \sum_{j=0}^{d-2} \frac{1}{(d-j)!} S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_j}$$

On the other hand, Lemma 2.10 also yields that there exists $q \in \mathcal{P}_{d-2}$ such that

$$\begin{aligned} \Delta_2 \left(2^{-d} \boldsymbol{v}_{p_{d-1}} + \boldsymbol{v}_{\widetilde{q}_{d-1}} \right) &= 2^{-d} \sum_{j=0}^{d-1} \frac{2^{d-k}}{(d-j)!} \boldsymbol{v}_{p_j} + \boldsymbol{v}_q \\ &= 2^{-d+1} \boldsymbol{v}_{p_{d-1}} + 2^{-d} \sum_{j=0}^{d-2} \frac{2^{d-k}}{(d-j)!} \boldsymbol{v}_{p_j} + \boldsymbol{v}_q. \end{aligned}$$

Substituting these two identities into (13), we thus obtain that

$$S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_{d-1}} = 2^{-d+1} \boldsymbol{v}_{p_{d-1}} + 2^{-d} \sum_{j=0}^{d-2} \frac{2^{d-k} - 1}{(d-j)!} \boldsymbol{v}_{p_j} + \boldsymbol{v}_q,$$

hence,

$$S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_{d-1}} = 2^{-d+1} \boldsymbol{v}_{p_{d-1}} + \boldsymbol{v}_{\widetilde{q}_{d-2}}, \qquad \widetilde{q}_{d-2} \in \mathcal{P}_{d-2}, \tag{14}$$

which is (13) with d replaced by $d-1$. Since the argument was only based on the stationarity of $S_{\boldsymbol{\mathcal{A}}}$, we can repeat this process and find that

$$S_{\boldsymbol{\mathcal{A}}} \boldsymbol{v}_{p_j} = 2^{-j} \boldsymbol{v}_{p_j} + \boldsymbol{v}_{\widetilde{q}_{j-1}}, \qquad q_j \in \mathcal{P}_{j-1}, \qquad j = 0, \dots, d. \tag{15}$$

Since $\mathcal{P}_j$ is spanned by $p_0, \dots, p_j$, (15) means that there exists an upper triangular matrix $\boldsymbol{U}$ with diagonal $1, \dots, 2^{-d+1}, 2^{-d}$ such that

$$S_{\boldsymbol{\mathcal{A}}} \left[\boldsymbol{v}_{p_0}, \dots, \boldsymbol{v}_{p_d} \right] = \left[\boldsymbol{v}_{p_0}, \dots, \boldsymbol{v}_{p_d} \right] \boldsymbol{U} = \left[\boldsymbol{v}_{p_0}, \dots, \boldsymbol{v}_{p_d} \right] \begin{bmatrix} 1 & * & \dots & * \\ & 2^{-1} & \ddots & \vdots \\ & & \ddots & * \\ & & & 2^{-d} \end{bmatrix}. \tag{16}$$

Using the upper triangular matrix $\boldsymbol{S}$ with diagonal elements 1 which factorizes $\boldsymbol{U}$ into

$$\boldsymbol{L} = \boldsymbol{S} \boldsymbol{D} \boldsymbol{S}^{-1}, \qquad \boldsymbol{D} = \begin{bmatrix} 1 & & & \\ & 2^{-1} & & \\ & & \ddots & \\ & & & 2^{-d} \end{bmatrix},$$

we now define

$$\left[\boldsymbol{v}_{\widehat{p}_0}, \dots, \boldsymbol{v}_{\widehat{p}_d} \right] := \left[\boldsymbol{v}_{p_0}, \dots, \boldsymbol{v}_{p_d} \right] \boldsymbol{S}$$

and obtain by a straightforward computation that

$$\begin{aligned} S_{\boldsymbol{\mathcal{A}}} \left[\boldsymbol{v}_{\widehat{p}_0}, \dots, \boldsymbol{v}_{\widehat{p}_d} \right] &= S_{\boldsymbol{\mathcal{A}}} \left[\boldsymbol{v}_{p_0}, \dots, \boldsymbol{v}_{p_d} \right] \boldsymbol{S} = \left[\boldsymbol{v}_{p_0}, \dots, \boldsymbol{v}_{p_d} \right] \boldsymbol{L} \boldsymbol{S} \\ &= \left[\boldsymbol{v}_{p_0}, \dots, \boldsymbol{v}_{p_d} \right] \boldsymbol{S} \boldsymbol{D} = \left[\boldsymbol{v}_{\widehat{p}_0}, \dots, \boldsymbol{v}_{\widehat{p}_d} \right] \boldsymbol{D}, \end{aligned}$$

hence $S_{\boldsymbol{\mathcal{A}}}$ satisfies the spectral condition of order d since $\boldsymbol{S}$ being upper triangular guarantees that any $\widehat{p}_j$ is of degree exactly j. $\square$

Corollary 2.12 *For a mask $\boldsymbol{\mathcal{A}} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ the following statements are equivalent:*

1. *$\boldsymbol{\mathcal{A}}$ satisfies the spectral condition of order d.*
2. *There exist masks $\boldsymbol{\mathcal{B}}_k$, $k = 0, \dots, d$ such that*

$$\begin{bmatrix} T_k & \\ & I \end{bmatrix} S_{\boldsymbol{\mathcal{A}}} = 2^{-k} S_{\boldsymbol{\mathcal{B}}_k} \begin{bmatrix} T_k & \\ & I \end{bmatrix}, \qquad S_{\boldsymbol{\mathcal{B}}_k} \boldsymbol{e}_k = \boldsymbol{e}_k.$$

3. *There exists* $\boldsymbol{\mathcal{B}}$ *such that*

$$T_d S_{\boldsymbol{\mathcal{A}}} = 2^{-d} S_{\boldsymbol{\mathcal{B}}} T_d, \qquad S_{\boldsymbol{\mathcal{B}}} \boldsymbol{e}_d = \boldsymbol{e}_d.$$

Proof: "1) $\Rightarrow$ 2)" has been the iterative step in [18] and the normalization is deduced as in Theorem 2.8, "2) $\Rightarrow$ 3)" is trivial and "3) $\Rightarrow$ 1)" is the statement of Theorem 2.11. $\square$

2.2 Convergence

Definition 2.13 *Let* $\boldsymbol{\mathcal{A}} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ *be a mask and* $H_{\boldsymbol{\mathcal{A}}}$ *the associated Hermite subdivision scheme on* $\ell^{d+1}(\mathbb{Z})$ *as defined in* (6). *The scheme is called* convergent *if for any data* $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$ *and the corresponding sequence of refinements* $\boldsymbol{f}_n$, *there exists a* ***vector*** *function* $\Phi = [\phi_i]_{i=0,\dots,d} \in C\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$ *such that for any compact* $K \subset \mathbb{R}$ *there exists a sequence* ε_n *with limit* 0 *which satisfies*

$$\max_{i=0,\dots,d} \max_{\alpha\in\mathbb{Z}\cap 2^n K} \left| f_n^{(i)}(\alpha) - \phi_i\left(2^{-n}\alpha\right)\right| \le \varepsilon_n. \tag{17}$$

The scheme $H_{\boldsymbol{\mathcal{A}}}$ *is said to be* C^ℓ–convergent *if moreover* $\phi_0 \in C^\ell(\mathbb{R},\mathbb{R})$ *and*

$$\frac{d^i\phi_0}{dx^i} = \phi_i, \quad i = 0,\dots,\min\{\ell, d\}.$$

We recall the following result on convergence proved in [18].

Theorem 2.14 *Let* $\boldsymbol{\mathcal{A}} \in \ell^{d+1}(\mathbb{Z})$ *be a mask which satisfies the spectral condition of order at least d. Suppose that for any data* $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$ *and associated refinement sequence* $\boldsymbol{f}_n$ *of the Hermite scheme* $H_{\boldsymbol{\mathcal{A}}}$,

1. *the sequence* $\boldsymbol{f}_n(0)$ *converges to a limit* $\boldsymbol{y} \in \mathbb{R}^{d+1}$,

2. *(at least) one of the following two properties holds true:*

   (a) *the associated* Taylor subdivision scheme $S_{\boldsymbol{\mathcal{B}}}$ *is* C^0*–convergent and for any initial data* $\boldsymbol{g}_0 = T_d\boldsymbol{f}_0$, *the limit function* $\Psi = \Psi_{\boldsymbol{g}} \in C\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$ *satisfies*

$$\Psi = \begin{bmatrix} \mathbf{0} \\ \psi_d \end{bmatrix}, \ \psi_d \in C(\mathbb{R},\mathbb{R}). \tag{18}$$

(b) *the associated* complete Taylor subdivision scheme $S_{\widetilde{\boldsymbol{\mathcal{B}}}}$ *is* contractive, *that is, it is C^0–convergent and for any initial data $\boldsymbol{g}_0 = \widetilde{T}_d \boldsymbol{f}_0$, the limit function $\Psi = \Psi_{\boldsymbol{g}} \in C\left(\mathbb{R}, \mathbb{R}^{d+1}\right)$ satisfies $\Psi = \mathbf{0}$.*

Then $H_{\boldsymbol{\mathcal{A}}}$ is C^d–convergent.

We conclude the section with an important corollary whose proof follows the lines of the proof of [18, Corollary 5].

Corollary 2.15 *Let $\boldsymbol{\mathcal{A}} \in \ell^{d+1}(\mathbb{Z})$ be a given mask which satisfies the spectral condition of order at least d. Suppose that the associated Taylor subdivision scheme $S_{\boldsymbol{\mathcal{B}}}$ is C^k–convergent with the conditions of Theorem 2.14, then $H_{\boldsymbol{\mathcal{A}}}$ is C^{d+k}-convergent.*

3 Extended schemes

With a Hermite scheme of degree d, we can get derivatives up to degree d and at most C^d–convergence. Sometimes, the last component of the limit function, namely $\phi_0^{(d)}$, turns out to be more regular than C^0. To explore this extra regularity, we propose to extend the scheme to one of higher degree. As examples, we explicitly study the de Rham transform and Cardinal splines that provide such regularity.

3.1 A first extension

From a given Hermite scheme with mask $\boldsymbol{\mathcal{A}}$, we first build a new scheme $\boldsymbol{\mathcal{A}}_+$, called *extended scheme*, by only adding a single row in an intuitive way. Its definition is based on the approximation of a derivative by a slope. Precisely for a function $\varphi \in C^{d+1}(\mathbb{R})$ and $\alpha \in \mathbb{Z}$, we use the fact that

$$\begin{aligned} \varphi^{(d+1)}\left(\frac{2\alpha}{2^{n+1}}\right) = \varphi^{(d+1)}\left(\frac{\alpha}{2^n}\right) &\simeq 2^{n-1}\left(\varphi^{(d)}\left(\frac{\alpha+1}{2^n}\right) - \varphi^{(d)}\left(\frac{\alpha-1}{2^n}\right)\right), \\ \varphi^{(d+1)}\left(\frac{2\alpha+1}{2^{n+1}}\right) &\simeq 2^{n}\left(\varphi^{(d)}\left(\frac{\alpha+1}{2^n}\right) - \varphi^{(d)}\left(\frac{\alpha}{2^n}\right)\right). \end{aligned}$$

Starting from any $\boldsymbol{f}_0$, the previous approximations suggest the construction for the additional components $f_{n+1}^{(d+1)}$ as

$$\begin{aligned} f_{n+1}^{(d+1)}(2\alpha) &= 2^{n-1}\left(f_n^{(d)}(\alpha+1) - f_n^{(d)}(\alpha-1)\right), \\ f_{n+1}^{(d+1)}(2\alpha+1) &= 2^{n}\left(f_n^{(d)}(\alpha+1) - f_n^{(d)}(\alpha)\right). \end{aligned}$$

The new mask is then defined in the usual way, with just d replaced by $d+1$ so that $\boldsymbol{A}_+(\alpha) \in \mathbb{R}^{(d+2)\times(d+2)}$ has the entries

$$\begin{aligned}
\boldsymbol{A}_+(-2) &= \left[\begin{array}{c|c} \boldsymbol{A}(-2) & \boldsymbol{0} \\ \hline \boldsymbol{0} \;\; 2^{-d-2} & 0 \end{array}\right], & \boldsymbol{A}_+(-1) &= \left[\begin{array}{c|c} \boldsymbol{A}(-1) & \boldsymbol{0} \\ \hline \boldsymbol{0} \;\; 2^{-d-1} & 0 \end{array}\right] \\
\boldsymbol{A}_+(1) &= \left[\begin{array}{c|c} \boldsymbol{A}(1) & \boldsymbol{0} \\ \hline \boldsymbol{0} \;\; -2^{-d-1} & 0 \end{array}\right], & \boldsymbol{A}_+(2) &= \left[\begin{array}{c|c} \boldsymbol{A}(2) & \boldsymbol{0} \\ \hline \boldsymbol{0} \;\; -2^{-d-2} & 0 \end{array}\right] \\
\boldsymbol{A}_+(\alpha) &= \left[\begin{array}{c|c} \boldsymbol{A}(\alpha) & \boldsymbol{0} \\ \hline \boldsymbol{0} & 0 \end{array}\right] & & \text{for } \alpha \notin \{-2,-1,1,2\}.
\end{aligned} \tag{19}$$

Proposition 3.1 *Suppose that the scheme with mask $\boldsymbol{\mathcal{A}}$ of degree d satisfies the spectral condition of order $d+1$ with the polynomials $p_0, p_1, \ldots, p_{d+1}$. Then the extension with mask $\boldsymbol{A}_+(\alpha)$ defined by* (19) *also satisfies the spectral condition of order $d+1$ with the same polynomials, thus there exist $\boldsymbol{\mathcal{B}}_+$ and $\widetilde{\boldsymbol{\mathcal{B}}}_+$ such that $T_{d+1} S_{\boldsymbol{\mathcal{A}}_+} = 2^{-d-1} S_{\boldsymbol{\mathcal{B}}_+} T_{d+1}$ and $\widetilde{T}_{d+1} S_{\boldsymbol{\mathcal{A}}_+} = 2^{-d-1} S_{\widetilde{\boldsymbol{\mathcal{B}}}_+} \widetilde{T}_{d+1}$.*

Proof: Since $\boldsymbol{\mathcal{A}}$ satisfies the spectral condition of order $d+1$, we obtain for any $i \in \{0, \ldots, d+1\}$ and any real numbers x, y and u that

$$p_i^{(d)}(x+u) - p_i^{(d)}(x) = u p_i^{(d+1)}(y) = \begin{cases} 0 & \text{for} \quad i = 0, \ldots, d, \\ u & \text{for} \quad i = d+1, \end{cases}$$

since $p_i \in \mathcal{P}_i$ and $p_{d+1}(x) = x^{d+1}/d! + \ldots$

If we define $\boldsymbol{v}_i(x) = [p_i(x), \ldots, p_i^{(d)}(x), p_i^{(d+1)}(x)]^T$ for $i = 0, \ldots, d+1$ then

$$\begin{aligned}
& S_{\boldsymbol{\mathcal{A}}_+} \boldsymbol{v}_i(2\alpha) \\
&= \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}_+(2\alpha - 2\beta) \boldsymbol{v}_i(\beta) = \sum_{\beta\in\mathbb{Z}} \begin{bmatrix} \boldsymbol{A}(2\alpha-2\beta) \begin{bmatrix} p_i(\beta) \\ \vdots \\ p_i^{(d)}(\beta) \end{bmatrix} \\ a_{d+1,d}(2\alpha-2\beta) p_i^{(d)}(\beta) \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{1}{2^i} \begin{bmatrix} p_i(2\alpha) \\ \vdots \\ p_i^{(d)}(2\alpha) \end{bmatrix} \\ 1/2^{d+2} \left(p_i^{(d)}(\alpha+1) - p_i^{(d)}(\alpha-1) \right) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2^i} \begin{bmatrix} p_i(2\alpha) \\ \vdots \\ p_i^{(d)}(2\alpha) \end{bmatrix} \\ 1/2^{d+1} p_i^{(d+1)}(2\alpha) \end{bmatrix} \\
&= \frac{1}{2^i} \boldsymbol{v}_i(2\alpha).
\end{aligned}$$

Similarly, $S_{\boldsymbol{\mathcal{A}}_+}\boldsymbol{v}_i(2\alpha+1) = \dfrac{1}{2^i}\boldsymbol{v}_i(2\alpha+1)$ and therefore the spectral condition of order $d+1$ is satisfied by the extended scheme $\boldsymbol{\mathcal{A}}_+$. Consequently, we can compute the masks $\boldsymbol{\mathcal{B}}_+$ and $\widetilde{\boldsymbol{\mathcal{B}}_+}$ such that $T_{d+1}S_{\boldsymbol{\mathcal{A}}_+} = 2^{-d-1}S_{\boldsymbol{\mathcal{B}}_+}T_{d+1}$ and $\widetilde{T}_{d+1}S_{\boldsymbol{\mathcal{A}}_+} = 2^{-d-1}S_{\widetilde{\boldsymbol{\mathcal{B}}_+}}\widetilde{T}_{d+1}$. $\square$

Examples of such extensions are given in the next section.

3.2 Factorization of extensions

We now suppose that $\boldsymbol{\mathcal{A}} \in \ell^{(d+1)\times(d+1)}(\mathbb{Z})$ is a mask which satisfies the spectral condition of order $d' > d$ and want to consider the general question of how to extend this mask in such a way that the extended scheme $\boldsymbol{\mathcal{A}}' \in \ell^{(d'+1)\times(d'+1)}(\mathbb{Z})$ satisfies a spectral condition of increased degree d'. We begin with a simple observation concerning the Taylor operators, namely,

$$\boldsymbol{\mathcal{T}}^*_{d'}(z) = \begin{bmatrix} \widetilde{\boldsymbol{\mathcal{T}}}^*_d(z) & -\boldsymbol{W} \\ \boldsymbol{0} & \boldsymbol{\mathcal{T}}^*_{\bar{d}}(z) \end{bmatrix}, \qquad \widetilde{\boldsymbol{\mathcal{T}}}^*_{d'}(z) = \begin{bmatrix} \widetilde{\boldsymbol{\mathcal{T}}}^*_d(z) & -\boldsymbol{W} \\ \boldsymbol{0} & \widetilde{\boldsymbol{\mathcal{T}}}^*_{\bar{d}}(z) \end{bmatrix}, \tag{20}$$

where $\bar{d} := d' - d - 1$ and

$$\boldsymbol{W} = \begin{bmatrix} \frac{1}{(d+1)!} & \cdots & \frac{1}{(d+\bar{d}+1)!} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & \frac{1}{(\bar{d}+1)!} \end{bmatrix} \in \mathbb{R}^{(d+1)\times(\bar{d}+1)}.$$

From (20) one can verify by straightforward computation that

$$\begin{aligned} \boldsymbol{\mathcal{T}}^*_{d'}(z)^{-1} &= \begin{bmatrix} \widetilde{\boldsymbol{\mathcal{T}}}^*_d(z)^{-1} & \widetilde{\boldsymbol{\mathcal{T}}}^*_d(z)^{-1}\,\boldsymbol{W}\,\boldsymbol{\mathcal{T}}^*_{\bar{d}}(z)^{-1} \\ \boldsymbol{0} & \boldsymbol{\mathcal{T}}^*_{\bar{d}}(z)^{-1} \end{bmatrix}, \\ \widetilde{\boldsymbol{\mathcal{T}}}^*_{d'}(z)^{-1} &= \begin{bmatrix} \widetilde{\boldsymbol{\mathcal{T}}}^*_d(z)^{-1} & \widetilde{\boldsymbol{\mathcal{T}}}^*_d(z)^{-1}\,\boldsymbol{W}\,\widetilde{\boldsymbol{\mathcal{T}}}^*_{\bar{d}}(z)^{-1} \\ \boldsymbol{0} & \widetilde{\boldsymbol{\mathcal{T}}}^*_{\bar{d}}(z)^{-1} \end{bmatrix}, \end{aligned} \tag{21}$$

respectively. Since $\det \boldsymbol{\mathcal{T}}^*_d(z) = (z^{-1}-1)^d$ and $\det \widetilde{\boldsymbol{\mathcal{T}}}^*_d(z) = (z^{-1}-1)^{d+1}$, the matrices in (21) are well–defined for $z \neq 1$. Now we extend $\boldsymbol{\mathcal{A}}$ to the matrix mask $\boldsymbol{\mathcal{A}}'$ by extending the symbol into the block lower triangular

$$\boldsymbol{\mathcal{A}}'^*(z) := \begin{bmatrix} \boldsymbol{\mathcal{A}}^*(z) & 0 \\ \boldsymbol{\mathcal{U}}^*(z) & \boldsymbol{\mathcal{V}}^*(z) \end{bmatrix}, \quad \boldsymbol{\mathcal{U}} \in \ell^{(\bar{d}+1)\times(d+1)}(\mathbb{Z}),\ \boldsymbol{\mathcal{V}} \in \ell^{(\bar{d}+1)\times(\bar{d}+1)}(\mathbb{Z}), \tag{22}$$

and compute

$$\begin{aligned}
\boldsymbol{\mathcal{T}}_{d'}^*(z)\boldsymbol{\mathcal{A}}'^*(z) &= \begin{bmatrix} \widetilde{\boldsymbol{\mathcal{T}}}_d^*(z)\boldsymbol{\mathcal{A}}^*(z) - \boldsymbol{W}\boldsymbol{\mathcal{U}}^*(z) & -\boldsymbol{W}\boldsymbol{\mathcal{V}}^*(z) \\ \boldsymbol{\mathcal{T}}_{\bar d}^*(z)\boldsymbol{\mathcal{U}}^*(z) & \boldsymbol{\mathcal{T}}_{\bar d}^*(z)\,\boldsymbol{\mathcal{V}}^*(z) \end{bmatrix} \\
&= \begin{bmatrix} \widetilde{\boldsymbol{\mathcal{B}}}^*(z)\widetilde{\boldsymbol{\mathcal{T}}}_d^*(z^2) - \boldsymbol{W}\boldsymbol{\mathcal{U}}^*(z) & -\boldsymbol{W}\boldsymbol{\mathcal{V}}^*(z) \\ \boldsymbol{\mathcal{T}}_{\bar d}^*(z)\boldsymbol{\mathcal{U}}^*(z) & \boldsymbol{\mathcal{T}}_{\bar d}(z)^*\,\boldsymbol{\mathcal{V}}^*(z) \end{bmatrix}.
\end{aligned}$$

Choosing $\boldsymbol{\mathcal{U}}^*(z) := \widetilde{\boldsymbol{\mathcal{U}}}^*(z)\,\boldsymbol{\mathcal{T}}_d^*(z^2)$ and $\boldsymbol{\mathcal{V}}^*(z) := \widetilde{\boldsymbol{\mathcal{U}}}^*(z)\boldsymbol{W} + \widetilde{\boldsymbol{\mathcal{V}}}^*(z)$, we get the simplified expression

$$\begin{aligned}
&\boldsymbol{\mathcal{T}}_{d'}^*(z)\,\boldsymbol{\mathcal{A}}'^*(z)\,\boldsymbol{\mathcal{T}}_{d'}^*(z^2)^{-1} \\
&= \begin{bmatrix} \widetilde{\boldsymbol{\mathcal{B}}}^*(z) - \boldsymbol{W}\widetilde{\boldsymbol{\mathcal{U}}}^*(z) & \left(\widetilde{\boldsymbol{\mathcal{B}}}^*(z)\,\boldsymbol{W} - \boldsymbol{W}\,\widetilde{\boldsymbol{\mathcal{U}}}^*(z)\,\boldsymbol{W} - \boldsymbol{W}\,\boldsymbol{\mathcal{V}}^*(z)\right)\boldsymbol{\mathcal{T}}_{\bar d}^*(z^2)^{-1} \\ \boldsymbol{\mathcal{T}}_{\bar d}^*(z)\widetilde{\boldsymbol{\mathcal{U}}}^*(z) & \boldsymbol{\mathcal{T}}_{\bar d}^*(z)\left(\widetilde{\boldsymbol{\mathcal{U}}}^*(z)\boldsymbol{W} + \boldsymbol{\mathcal{V}}^*(z)\right)\boldsymbol{\mathcal{T}}_{\bar d}^*(z^2)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \widetilde{\boldsymbol{\mathcal{B}}}^*(z) - \boldsymbol{W}\widetilde{\boldsymbol{\mathcal{U}}}^*(z) & \left(\widetilde{\boldsymbol{\mathcal{B}}}^*(z)\,\boldsymbol{W} - \boldsymbol{W}\,\widetilde{\boldsymbol{\mathcal{V}}}^*(z)\right)\boldsymbol{\mathcal{T}}_{\bar d}^*(z^2)^{-1} \\ \boldsymbol{\mathcal{T}}_{\bar d}^*(z)\widetilde{\boldsymbol{\mathcal{U}}}^*(z) & \boldsymbol{\mathcal{T}}_{\bar d}^*(z)\,\widetilde{\boldsymbol{\mathcal{V}}}^*(z)\,\boldsymbol{\mathcal{T}}_{\bar d}^*(z^2)^{-1} \end{bmatrix}.
\end{aligned}$$

Thus, Corollary 2.12 yields the following conclusion which describes the algebraic condition on the extension that guarantees that the extended scheme satisfies the spectral condition of order d'.

Proposition 3.2 *The mask $\boldsymbol{\mathcal{A}}'$ defined by the extension* (22) *satisfies the spectral condition of order d', i.e., $T_{d'}S_{\boldsymbol{\mathcal{A}}'} = S_{\boldsymbol{\mathcal{B}}'}T_{d'}$ and $S_{\boldsymbol{\mathcal{B}}'}\boldsymbol{e}_{d'} = \boldsymbol{e}_{d'}$, if and only if*

1. *$\widetilde{\boldsymbol{\mathcal{V}}}$ satisfies the spectral condition of order $\bar d - 1$,*
2. *$\left(\widetilde{\boldsymbol{\mathcal{B}}}^*(z)\,\boldsymbol{W} - \boldsymbol{W}\,\widetilde{\boldsymbol{\mathcal{V}}}^*(z)\right)\boldsymbol{\mathcal{T}}_{\bar d}^*(z)^{-1}\boldsymbol{e}_{\bar d} = 0.$*

Since $T_{\bar d}\boldsymbol{e}_{\bar d} = \boldsymbol{e}_{\bar d}$, hence also $T_{\bar d}^{-1}\boldsymbol{e}_{\bar d} = \boldsymbol{e}_{\bar d}$, the second condition in Proposition 3.2 is equivalent to $\widetilde{\boldsymbol{\mathcal{B}}}^*(z)\,\boldsymbol{W}\boldsymbol{e}_{\bar d} = \boldsymbol{W}\,\widetilde{\boldsymbol{\mathcal{V}}}^*(z)\boldsymbol{e}_{\bar d}$ or

$$\boldsymbol{v}_{\bar d}(z) = \boldsymbol{W}^{-1}\widetilde{\boldsymbol{\mathcal{B}}}^*(z)\begin{bmatrix} \frac{1}{d'!} \\ \vdots \\ \frac{1}{(d'-d)!} \end{bmatrix}, \tag{23}$$

where $\boldsymbol{v}_{\bar d}$ denotes the last column of $\widetilde{\boldsymbol{\mathcal{V}}}^*$.

4 The Example of de Rham schemes

4.1 Construction and first properties

The construction of the de Rham transform was proposed in [7] and some of the properties used here can be found in [6]. In the de Rham scheme, a subdivision scheme is applied twice and then one only keeps the values at indices which are equal to 1 and 3 modulo 4, thus defining another binary subdivision scheme derived from the original one. With de Rham scheme, one loses the interpolating property but can gain higher order of regularity. The challenge is to prove this higher regularity of the limit function.

For a Hermite subdivision scheme $H_{\boldsymbol{\mathcal{A}}}$ of degree d, with mask $\{\boldsymbol{A}(\alpha)\}_{\alpha\in\mathbb{Z}}$, let us define a new (dual) Hermite subdivision scheme by taking de Rham transform of $\boldsymbol{\mathcal{A}}$. From the initial state of the scheme $\bar{\boldsymbol{f}}_0 = \boldsymbol{f}_0 : \mathbb{Z} \to \mathbb{R}^{d+1}$, we define the sequence $\bar{\boldsymbol{f}}_n$ for $n > 0$ by

$$\begin{aligned}
\boldsymbol{D}^{n+1}\boldsymbol{g}(\beta) &= \sum_{\gamma\in\mathbb{Z}} \boldsymbol{A}(\beta - 2\gamma)\boldsymbol{D}^n\bar{\boldsymbol{f}}_n(\gamma),\ \beta \in \mathbb{Z}\\
\boldsymbol{D}^{n+2}\boldsymbol{h}(\alpha) &= \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\alpha - 2\beta)\boldsymbol{D}^{n+1}\boldsymbol{g}(\beta),\ \alpha \in \mathbb{Z}\\
\bar{\boldsymbol{f}}_{n+1}(\alpha) &= \boldsymbol{h}(2\alpha + 1),\ \alpha \in \mathbb{Z}.
\end{aligned}$$

Then it is easy to prove that $\boldsymbol{D}^{n+1}\bar{\boldsymbol{f}}_{n+1}(\alpha) = \sum_{\gamma\in\mathbb{Z}} \bar{\boldsymbol{A}}(\alpha - 2\gamma)\boldsymbol{D}^n\bar{\boldsymbol{f}}_n(\gamma)$ where

$$\bar{\boldsymbol{A}}(\alpha) = \boldsymbol{D}^{-1}\sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(2\alpha + 1 - 2\beta)\boldsymbol{A}(\beta), \qquad \alpha \in \mathbb{Z}. \tag{24}$$

Definition 4.1 *Let $S_{\boldsymbol{\mathcal{A}}}$ be a subdivision scheme.* De Rham transform $S_{\overline{\boldsymbol{\mathcal{A}}}}$ of $S_{\boldsymbol{\mathcal{A}}}$ *is the subdivision scheme whose mask $\overline{\boldsymbol{\mathcal{A}}}$ is defined by* (24)*.*

Remark 4.2 *Note that if the support of $S_{\boldsymbol{\mathcal{A}}}$ is $[\sigma, \sigma']$, then the support of its de Rham transform $S_{\overline{\boldsymbol{\mathcal{A}}}}$ is contained in $[(3\sigma - 1)/2, (3\sigma' - 1)/2]$.*

The following Theorem and Corollary are proved in [6].

Theorem 4.3 *Let $S_{\boldsymbol{\mathcal{A}}}$ be a Hermite subdivision scheme which satisfies the spectral condition of order ℓ, then de Rham transform $S_{\overline{\boldsymbol{\mathcal{A}}}}$ satisfies the spectral condition $\sum_{\beta\in\mathbb{Z}} \bar{\boldsymbol{A}}(\alpha - 2\beta)\bar{\boldsymbol{v}}_{p_j}(\beta) = \bar{\boldsymbol{v}}_{p_j}(\alpha)/2^j$ for an appropriate sequence of polynomials $\bar{p}_j$ of degree j, $j = 0, \ldots, \ell$.*

Corollary 4.4 *Let there be a Hermite scheme of degree d with mask $\boldsymbol{\mathcal{A}}$ which reproduces a basis of $\mathcal{P}_\ell$, then its de Rham transform with mask $\overline{\boldsymbol{\mathcal{A}}}$ satisfies the corresponding spectral condition with the sequence of polynomials $\bar{p}_k(x) = (x - 1/2)^k/k!$, $k = 0, \ldots, \ell$, $\ell \leq d$.*

Remark 4.5 *From [10], we know that a convergent Hermite interpolatory scheme of degree d reproduces any polynomial of degree at most d.*

4.2 An extension of a Hermite subdivision scheme of degree 1

We start with the interpolatory Hermite subdivision scheme of degree $d = 1$ proposed in [16], depending on two parameters λ, μ. The non zero matrices of its mask are $\boldsymbol{A}(-1)$, $\boldsymbol{A}(0)$, $\boldsymbol{A}(1)$, given as

$$\frac{1}{4}\begin{bmatrix} 2 & 4\lambda \\ 2(1-\mu) & \mu \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix} 2 & -4\lambda \\ -2(1-\mu) & \mu \end{bmatrix}. \tag{25}$$

For any values of the parameters λ and μ, the scheme reproduces polynomials of degree 1. Moreover it reproduces $\mathcal{P}_2$ if and only if $\lambda = -1/8$ and $\mathcal{P}_3$ if also $\mu = -0.5$.

Applying de Rham transform, we end up with the dual approximating Hermite subdivision scheme $\bar{\boldsymbol{\mathcal{A}}}$. The non zero entries of the mask, $\bar{\boldsymbol{A}}(-2)$, $\bar{\boldsymbol{A}}(-1)$, $\bar{\boldsymbol{A}}(0)$, $\bar{\boldsymbol{A}}(1)$, are computed as (see also [18])

$$\frac{1}{8}\begin{bmatrix} 2+4\lambda(1-\mu) & 4\lambda+2\lambda\mu \\ 4-2\mu-2\mu^2 & \mu^2+8\lambda(1-\mu) \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 6-4\lambda(1-\mu) & 8\lambda-2\lambda\mu \\ 4-2\mu-2\mu^2 & \mu^2-8\lambda(1-\mu)+2\mu \end{bmatrix},$$

$$\frac{1}{8}\begin{bmatrix} 6-4\lambda(1-\mu) & -8\lambda+2\lambda\mu \\ -4+2\mu+2\mu^2 & \mu^2-8\lambda(1-\mu)+2\mu \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 2+4\lambda(1-\mu) & -4\lambda-2\lambda\mu \\ -4+2\mu+2\mu^2 & \mu^2+8\lambda(1-\mu) \end{bmatrix}, \tag{26}$$

respectively. Since $H_{\bar{\boldsymbol{\mathcal{A}}}}$ satisfies the spectral condition of order $\ell = 1$, we can construct masks $\boldsymbol{\mathcal{B}}$ and $\tilde{\boldsymbol{\mathcal{B}}}$ supported on $[-1, 1]$ such that

$$\begin{bmatrix} z^{-1}-1 & -1 \\ 0 & 1 \end{bmatrix} \bar{\boldsymbol{\mathcal{A}}}^*(z) = \frac{1}{2}\boldsymbol{\mathcal{B}}^*(z) \begin{bmatrix} z^{-2}-1 & -1 \\ 0 & 1 \end{bmatrix} \tag{27}$$

and

$$\begin{bmatrix} z^{-1}-1 & -1 \\ 0 & z^{-1}-1 \end{bmatrix} \bar{\boldsymbol{\mathcal{A}}}^*(z) = \frac{1}{2}\tilde{\boldsymbol{\mathcal{B}}}^*(z) \begin{bmatrix} z^{-2}-1 & -1 \\ 0 & z^{-2}-1 \end{bmatrix}. \tag{28}$$

From the above, we see that the mask $\tilde{\boldsymbol{\mathcal{B}}}$ is the difference mask of $\boldsymbol{\mathcal{B}}$, that is

$$\tilde{\boldsymbol{\mathcal{B}}}^*(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1}-1 \end{bmatrix} \boldsymbol{\mathcal{B}}^*(z)$$

The non-zero entries of mask $\tilde{\boldsymbol{\mathcal{B}}}$ are

$$\tilde{\boldsymbol{B}}(-1) = \frac{1}{4}\begin{bmatrix} 2+4\lambda(1-\mu) & 2\lambda(1+\mu) \\ 4-2\mu-2\lambda^2 & \mu^2+8\lambda(1-\mu) \end{bmatrix}$$
$$\tilde{\boldsymbol{B}}(0) = \frac{1}{4}\begin{bmatrix} 2\mu+2\mu^2-8\lambda(1-\mu) & -4\lambda(1-\mu)-\mu^2 \\ 0 & 2\mu-16\lambda(1-\mu) \end{bmatrix},$$
$$\tilde{\boldsymbol{B}}(1) = \frac{1}{4}\begin{bmatrix} -2+4\lambda(1-\mu)+2\mu+2\mu^2 & -6\lambda\mu-\mu^2-2\mu+2 \\ -4+2\mu+2\mu^2 & 4-2\mu-\mu^2+8\lambda(1-\mu) \end{bmatrix}. \tag{29}$$

For some values of the parameters, the Hermite scheme $H_{\bar{\boldsymbol{\mathcal{A}}}}$ has C^1–convergence. Our aim is to show that the subdivision scheme associated with $\bar{\boldsymbol{\mathcal{A}}}$ can even be C^2 for an appropriate choice of parameters. To that end, we build $\bar{\boldsymbol{\mathcal{A}}}_+$ and show that this is C^2 by proving that $\widetilde{\boldsymbol{\mathcal{B}}}_+$ is contractive.

For our case, we fix $\lambda = -1/8$ so that $\boldsymbol{\mathcal{A}}$ reproduces any polynomial of $\mathcal{P}_2$. Using Corollary 4.4, de Rham scheme $\bar{\boldsymbol{\mathcal{A}}}$ satisfies the spectral condition of order $\ell = 2$ with the polynomials $\bar{p}_0(x) = 1$, $\bar{p}_1(x) = x - 1/2$ and $\bar{p}_2(x) = 1/2(x-1/2)^2$ and so does $\bar{\boldsymbol{\mathcal{A}}}_+$ from (19). The computation of $\widetilde{\bar{\boldsymbol{\mathcal{B}}}}_+$ gives

$$\widetilde{\bar{\boldsymbol{B}}_+}(-1) = \frac{1}{8}\begin{bmatrix} 6+2\mu & -2-\mu & 0 \\ 16-8\mu-8\mu^2 & -4+4\mu+4\mu^2 & 0 \\ 0 & 4 & 0 \end{bmatrix},$$
$$\widetilde{\bar{\boldsymbol{B}}_+}(0) = \frac{1}{4}\begin{bmatrix} 2+2\mu+4\mu^2 & -\mu-2\mu^2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$
$$\widetilde{\bar{\boldsymbol{B}}_+}(1) = \frac{1}{8}\begin{bmatrix} -10+10\mu+8\mu^2 & 4-5\mu-4\mu^2 & 1 \\ -16+8\mu+8\mu^2 & 4-4\mu-4\mu^2 & 4 \\ 0 & -4 & 4 \end{bmatrix}, \tag{30}$$
$$\widetilde{\bar{\boldsymbol{B}}_+}(2) = \frac{1}{4}\begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -2 & 2 \end{bmatrix}$$

Now, following [1], to prove the contractivity of the scheme with mask $\widetilde{\bar{\mathcal{B}}_+}$, we build the two matrices

$$\boldsymbol{U}_0 = \begin{bmatrix} \widetilde{\bar{\boldsymbol{B}}_+}(2) & \widetilde{\bar{\boldsymbol{B}}_+}(0) & \boldsymbol{0} \\ \boldsymbol{0} & \widetilde{\bar{\boldsymbol{B}}_+}(1) & \widetilde{\bar{\boldsymbol{B}}_+}(-1) \\ \boldsymbol{0} & \widetilde{\bar{\boldsymbol{B}}_+}(2) & \widetilde{\bar{\boldsymbol{B}}_+}(0) \end{bmatrix}, \quad \boldsymbol{U}_1 = \begin{bmatrix} \widetilde{\bar{\boldsymbol{B}}_+}(1) & \widetilde{\bar{\boldsymbol{B}}_+}(-1) & \boldsymbol{0} \\ \widetilde{\bar{\boldsymbol{B}}_+}(2) & \widetilde{\bar{\boldsymbol{B}}_+}(0) & \boldsymbol{0} \\ \boldsymbol{0} & \widetilde{\bar{\boldsymbol{B}}_+}(1) & \widetilde{\bar{\boldsymbol{B}}_+}(-1) \end{bmatrix},$$

and compute, for a given positive integer p, all the matrices

$$\boldsymbol{M}_{\boldsymbol{u}} = \prod_{k=1}^{p} \boldsymbol{U}_0^{u_k} \boldsymbol{U}_1^{1-u_k}, \qquad u_k \in \{0,1\}, \qquad \boldsymbol{u} := [u_k]_{k=1,\dots,p},$$

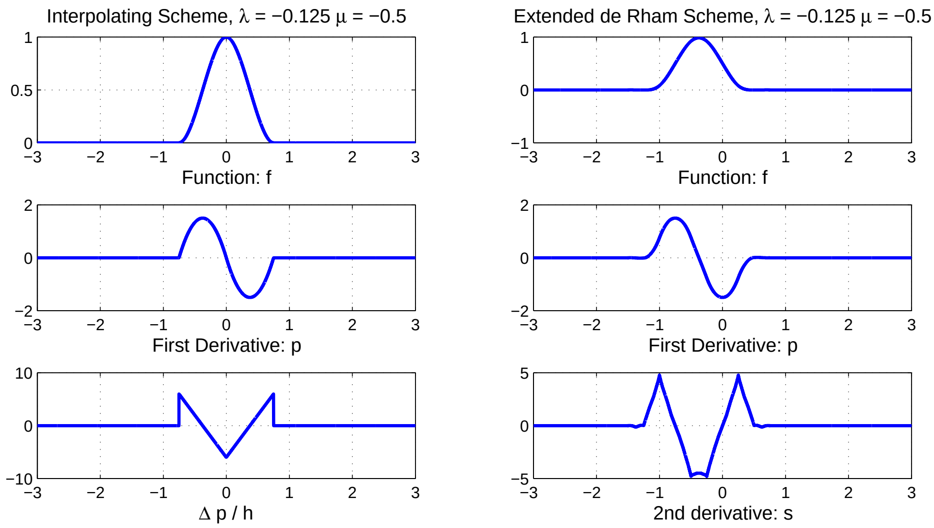

Figure 1: Interpolating and Extended de Rham Schemes

and finally the value $\rho_p = \max_{\boldsymbol{u}} \|\boldsymbol{M}_{\boldsymbol{u}}\|_1^{1/p}$. If $\rho_p < 1$ then the scheme $S_{\widetilde{\mathcal{B}}_+}$ is contractive and the Hermite scheme $H_{\widetilde{\mathcal{A}}_+}$ is C^2. See Figures 1 and 2 where the C^2–convergence is obtained for $\mu \in [-0.8, 0.37]$. .

4.3 An extension of a Hermite subdivision scheme of degree 2

In this second case, we start with the interpolatory Hermite subdivision scheme of degree $d = 2$ proposed in [16]. This is a 2 point subdivision scheme depending on several parameters. The non zero matrices of its mask are $\boldsymbol{A}(-1)$, $\boldsymbol{A}(0)$, $\boldsymbol{A}(1)$, given as

$$\boldsymbol{D}\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{bmatrix}, \quad \boldsymbol{D}, \quad \boldsymbol{D}\begin{bmatrix}\alpha_1 & -\alpha_2 & \alpha_3\\ -\beta_1 & \beta_2 & -\beta_3\\ \gamma_1 & -\gamma_2 & \gamma_3\end{bmatrix}, \qquad \boldsymbol{D} := \begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{4}\end{bmatrix}. \tag{31}$$

To guarantee reproduction of degree 3 polynomials, among the free parameters, the following constraints are assumed (see [13]):

$$\begin{aligned}\alpha_1 &= \tfrac{1}{2}, \quad \gamma_1 = 0, \quad \beta_2 = \tfrac{1-\beta_1}{2}, \quad \gamma_3 = \tfrac{1-\gamma_2}{2},\\ \alpha_3 &= \tfrac{-1-8\alpha_2}{16}, \ \beta_3 = \tfrac{2\beta_1-3}{24}.\end{aligned} \tag{32}$$

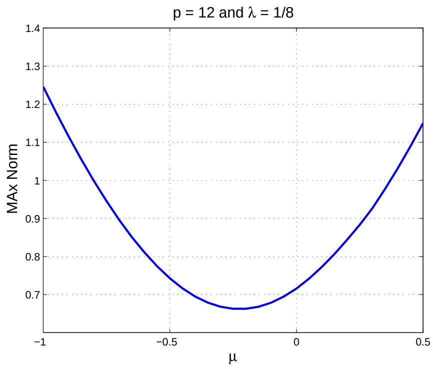

Figure 2: C^2–convergence when $MaxNorm < 1$.

Since the mask under consideration depends only on $\alpha_2,\ \beta_1,\ \gamma_2$, we rename the free parameters for the sake of brevity as α, β, γ, respectively, and denote by $\mathcal{R}$ the range of variation for them generating a C^2 scheme given in [13]. Therefore, the mask we consider reads as

$$\begin{bmatrix} \frac{1}{2} & \alpha & \frac{-1-8\alpha}{16} \\ \frac{\beta}{2} & \frac{1-\beta}{4} & \frac{2\beta-3}{48} \\ 0 & \frac{\gamma}{4} & \frac{1-\gamma}{8} \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & -\alpha & \frac{-1-8\alpha}{16} \\ -\frac{\beta}{2} & \frac{1-\beta}{4} & -\frac{2\beta-3}{48} \\ 0 & -\frac{\gamma}{4} & \frac{1-\gamma}{8} \end{bmatrix}, \ (\alpha,\beta,\gamma) \in \mathcal{R} \tag{33}$$

We apply de Rham transform for the dual approximating Hermite subdivision scheme $\bar{\boldsymbol{\mathcal{A}}}$. The non zero entries of the mask are given by

$$\bar{\boldsymbol{A}}(-2) = \begin{bmatrix} \frac{1+2\alpha\beta}{4} & \frac{\alpha(3-\beta)}{4} - \frac{(1+8\alpha)\gamma}{64} & \frac{\alpha(2\beta-3)}{48} - \frac{(1+8\alpha)(5-\gamma)}{128} \\ \frac{(3-\beta)\beta}{4} & \alpha\beta + \frac{(1-\beta)^2}{8} + \frac{\gamma(2\beta-3)}{96} & -\frac{\beta(1+8\alpha)}{16} + \frac{(3-2\beta-\gamma)(2\beta-3)}{192} \\ \frac{\beta\gamma}{2} & \frac{(3-2\beta-\gamma)\gamma}{8} & \frac{(2\beta-3)\gamma}{48} + \frac{(1-\gamma)^2}{16} \end{bmatrix}$$

$$\bar{\boldsymbol{A}}(-1) = \begin{bmatrix} \frac{3-2\alpha\beta}{4} & \frac{\alpha(3+\beta)}{4} - \frac{(1+8\alpha)\gamma}{64} & -\frac{\alpha(2\beta-3)}{48} - \frac{(1+8\alpha)(7-\gamma)}{128} \\ \frac{(3-\beta)\beta}{4} & -\alpha\beta + \frac{(1-\beta)(3-\beta)}{8} - \frac{(2\beta-3)\gamma}{96} & \frac{\beta(1+8\alpha)}{16} + \frac{(3-2\beta+\gamma)(2\beta-3)}{192} \\ -\frac{\beta\gamma}{2} & \frac{(3+2\beta-\gamma)\gamma}{8} & -\frac{(2\beta-3)\gamma}{48} + \frac{(1-\gamma)(3-\gamma)}{16} \end{bmatrix}$$

$$\bar{\boldsymbol{A}}(0) = \begin{bmatrix} \frac{3-2\alpha\beta}{4} & -\frac{\alpha(3+\beta)}{4} + \frac{\gamma(1+8\alpha)}{64} & -\frac{\alpha(2\beta-3)}{48} - \frac{(1+8\alpha)(7-\gamma)}{128} \\ \frac{(\beta-3)\beta}{4} & -\alpha\beta + \frac{(1-\beta)(3-\beta)}{8} - \frac{(2\beta-3)\gamma}{96} & -\frac{\beta(1+8\alpha)}{16} + \frac{(2\beta-3-\gamma)(2\beta-3)}{192} \\ -\frac{\beta\gamma}{2} & \frac{(\gamma-3-2\beta)\gamma}{8} & -\frac{(2\beta-3)\gamma}{48} + \frac{(1-\gamma)(3-\gamma)}{16} \end{bmatrix}$$

$$\bar{\boldsymbol{A}}(1) = \begin{bmatrix} \frac{1+2\alpha\beta}{4} & \frac{\alpha(\beta-3)}{4} + \frac{(1+8\alpha)\gamma}{64} & +\frac{\alpha(2\beta-3)}{48} - \frac{(1+8\alpha)(5-\gamma)}{128} \\ \frac{(\beta-3)\beta}{4} & \alpha\beta + \frac{(1-\beta)^2}{8} + \frac{\gamma(2\beta-3)}{96} & \frac{(1+8\alpha)\beta}{16} - \frac{(2\beta-3)(3-2\beta-\gamma)}{192} \\ \frac{\beta\gamma}{2} & \frac{\gamma(2\beta+\gamma-3)}{8} & \frac{(2\beta-3)\gamma}{48} + \frac{(1-\gamma)^2}{16} \end{bmatrix}$$

Now, the parameters for the interpolating scheme $\boldsymbol{\mathcal{A}}$ have been chosen in such a way that it reproduces polynomials in $\mathcal{P}_3$. Again, using Corollary 4.4, de Rham scheme $\bar{\boldsymbol{\mathcal{A}}}$ satisfies the spectral condition of order 3 with the polynomials $\bar{p}_0(x) = 1$, $\bar{p}_1(x) = x - 1/2$, $\bar{p}_2(x) = 1/2(x-1/2)^2$ and $\bar{p}_3(x) = 1/6(x-1/2)^3$ and so does the extended scheme $\bar{\boldsymbol{\mathcal{A}}}_+$.

We recall that

$$\widetilde{\boldsymbol{T}}_3^*(z) = \begin{bmatrix} z^{-1}-1 & -1 & \frac{-1}{2} & \frac{-1}{6} \\ 0 & z^{-1}-1 & -1 & \frac{-1}{2} \\ 0 & 0 & z^{-1}-1 & -1 \\ 0 & 0 & 0 & z^{-1}-1 \end{bmatrix},$$

and the computation of $\widetilde{\bar{\boldsymbol{\mathcal{B}}}}_+$ satisfying $\widetilde{T}_3 S_{\boldsymbol{\mathcal{A}}} = 2^{-3} S_{\widetilde{\bar{\boldsymbol{\mathcal{B}}}}_+} \widetilde{T}_3$ gives

$$\widetilde{\bar{\boldsymbol{B}}}(-1) = 8 \begin{bmatrix} \frac{1+2\alpha\beta}{4} & \frac{144\alpha-3\gamma-48\alpha\beta-24\alpha\gamma}{192} & -\frac{-16\alpha\beta-24\alpha\gamma+15-3\gamma+144\alpha}{384} & 0 \\ \frac{\beta(-\beta+3)}{4} & \frac{96\alpha\beta+12-24\beta+12\beta^2+2\beta\gamma-3\gamma}{96} & -\frac{96\alpha\beta+9+4\beta^2+2\beta\gamma-3\gamma}{192} & 0 \\ \frac{\beta\gamma}{2} & \frac{\gamma(3-2\beta-\gamma)}{8} & -\frac{-2\beta\gamma-3-3\gamma^2+9\gamma}{48} & 0 \\ 0 & 0 & \frac{1}{16} & 0 \end{bmatrix},$$

$$\widetilde{\bar{\boldsymbol{B}}}(0) = 8 \left[\begin{matrix} \frac{-4\alpha\beta-\beta\gamma+\beta^2-3\beta+2}{4} & \frac{-24\beta^2+48\beta-24-96\alpha\beta+12\gamma^2-30\gamma+20\beta\gamma}{192} & \cdots \\ -\frac{\beta\gamma}{2} & +\frac{-24\beta+24-192\alpha\beta-30\gamma+20\beta\gamma+12\gamma^2}{96} & \cdots \\ -\beta\gamma & \frac{4\gamma\beta}{8} & \cdots \\ 0 & 0 & \cdots \end{matrix} \right.$$

$$\left. \begin{matrix} \cdots & \frac{8\beta^2+30\gamma-4-4\beta\gamma+160\alpha\beta-12\gamma^2}{384} & 0 \\ \cdots & \frac{-12\gamma^2+24\beta+192\alpha\beta-4\beta\gamma+30\gamma-18}{192} & 0 \\ \cdots & \frac{3-4\beta\gamma}{48} & 0 \\ \cdots & \frac{1}{16} & 0 \end{matrix} \right],$$

$$\widetilde{\bar{\boldsymbol{B}}}(1) = 8 \left[\begin{matrix} \frac{2\alpha\beta+1+\beta^2-3\beta+\beta\gamma}{4} & \frac{144\alpha\beta+96\beta-24\beta^2-20\beta\gamma+24\alpha\gamma-39\gamma-24+12\gamma^2-144\alpha}{192} & \cdots \\ \frac{\beta(\beta-3+2\gamma)}{4} & \frac{-39\gamma+12+12\gamma^2+96\alpha\beta+48\beta-12\beta^2-22\beta\gamma}{96} & \cdots \\ \frac{\beta\gamma}{2} & \frac{\gamma(-3-2\beta+\gamma)}{8} & \cdots \\ 0 & 0 & \cdots \end{matrix} \right.$$

$$\left. \begin{matrix} \cdots & -\frac{24\alpha\gamma+48\beta+176\alpha\beta-8\beta^2-4\beta\gamma-7-144\alpha-39\gamma+12\gamma^2}{384} & \frac{1}{384} \\ \cdots & -\frac{-39\gamma+24\beta-6\beta\gamma+12\gamma^2+15+96\alpha\beta-4\beta^2}{192} & \frac{1}{64} \\ \cdots & -\frac{-9\gamma+3\gamma^2-2\beta\gamma+3}{48} & \frac{1}{16} \\ \cdots & -\frac{1}{16} & \frac{1}{16} \end{matrix} \right]$$

$$\widetilde{\bar{\boldsymbol{B}}}(2) = 8 \begin{bmatrix} 0 & 0 & -\frac{1}{96} & \frac{1}{96} \\ 0 & 0 & -\frac{1}{32} & \frac{1}{32} \\ 0 & 0 & -\frac{1}{16} & \frac{1}{16} \end{bmatrix}.$$

We use the method of the previous subsection to prove that de Rham scheme is C^3 see Figures 3 and 4.

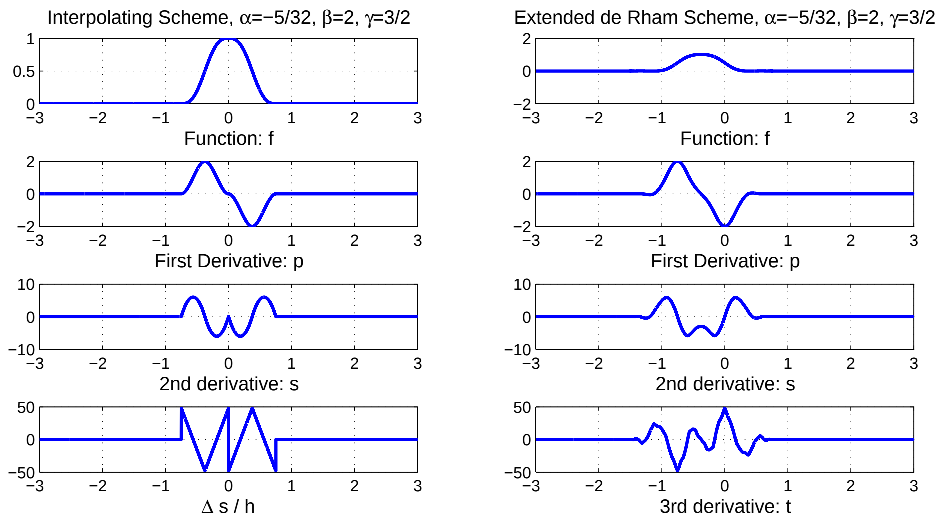

Figure 3: Interpolating and Extended de Rham Schemes

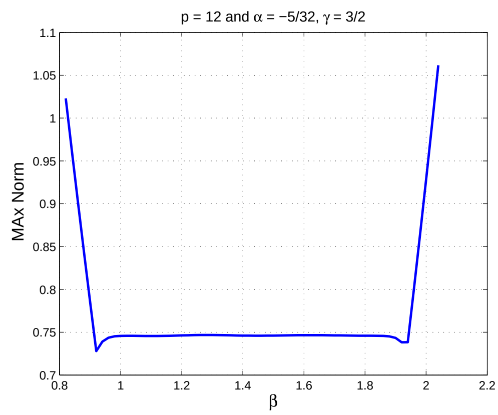

Figure 4: The C^3–convergence is obtained for $\beta \in [0.82, 2.05]$

5 Cardinal spline functions and Hermite schemes

In this section, we rewrite the well known cardinal splines (cf. [22]) in terms of a scalar subdivision scheme (or vector scheme of dimension 1 or Hermite scheme of degree 0). Then we extend this first subdivision scheme into Hermite schemes of different degrees.

5.1 Construction

Our presentation is based on a construction detailed by Micchelli in [20]. Let

$$\varphi_0(x) = \chi_{[0,1]} = \begin{cases} 1 \text{ if } x \in [0,1], \\ 0 \text{ if } x \notin [0,1]. \end{cases}$$

For $r = 1, 2, \ldots$, we build φ_r by means of autoconvolution as $\varphi_r = \varphi_0 * \varphi_{r-1}$ or $\varphi_r(x) = \int_{x-1}^{x} \varphi_{r-1}(t)dt$.

By recursion, it is easily seen that $\sigma_r := \text{supp}(\varphi_r) = [0, r+1]$, that φ_r is a $\mathcal{C}^{r-1}$ piecewise polynomial of degree r and that the translates of the functions $\varphi_r(\cdot - \alpha)$, $\alpha \in \mathbb{Z}$, form a nonnegative partition of unity, i.e. $\sum_{\alpha\in\mathbb{Z}} \varphi_r(x-\alpha) = 1$ and $\varphi_r(x-\alpha) \geq 0$. Moreover,

$$\varphi_r(x) = \frac{1}{2^r} \sum_{\alpha\in\mathbb{Z}} \binom{r+1}{\alpha} \varphi_r(2x-\alpha), \qquad \binom{i}{j} = \begin{cases} \frac{i!}{j!(i-j)!} & \text{if } 0 \leq j \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

Considering $v(x) = \sum_{\alpha\in\mathbb{Z}} f_0^{(0)}(\alpha)\varphi_r(x-\alpha)$, which is a finite sum for any $x \in \mathbb{R}$, we deduce for $n \in \mathbb{N}_0$ that $v(x) = \sum_{\alpha\in\mathbb{Z}} f_n^{(0)}(\alpha)\varphi_r\left(2^n x - \alpha\right)$ where

$$f_{n+1}^{(0)}(\alpha) = \frac{1}{2^r} \sum_{\beta\in\mathbb{Z}} \binom{r+1}{\alpha-2\beta} f_n^{(0)}(\beta) =: \sum_{\beta\in\mathbb{Z}} a_r(\alpha-2\beta) f_n^{(0)}(\beta), \qquad \alpha \in \mathbb{Z}, \tag{34}$$

that is,

$$a_r(\alpha) = \frac{1}{2^r}\binom{r+1}{\alpha}, \qquad \alpha \in \mathbb{Z}. \tag{35}$$

Moreover, the well–known derivative formula for cardinal B–spline yields

$$\frac{d^i v}{dx^i}(x) = \sum_{\alpha\in\mathbb{Z}} 2^{ni} \Delta^i f_n^{(0)}(\alpha - i)\, \varphi_{r-i}\left(2^n x - \alpha\right), \qquad i = 0, \ldots, r-1. \tag{36}$$

We have a particular case when $i = r-1$. Since the function φ_1 is piecewise linear with $\varphi_1(\alpha) = \delta_{1\alpha}$, we obtain $\dfrac{d^{r-1}v}{dx^{r-1}}(\beta/2^n) = 2^{ni}\Delta^i f_n^{(0)}(\beta - r + 1)$.

5.2 Spectral properties of S_{a_r}

We look for the eigenpolynomials of the scheme, namely the polynomials p such that $S_{a_r} p = \lambda p$. The following lemma was already given in [1] in the multidimensional case.

Lemma 5.1 *Let S_a be a scalar subdivision operator. If p and $q := S_a p$ are both polynomials, then*

$$q(x) = \frac{1}{2} \sum_{\beta \in \mathbb{Z}} a(\beta) p \left(\frac{x - \beta}{2} \right), \qquad x \in \mathbb{R}. \tag{37}$$

We also recall the following simple fact that can be proved by standard subdivision techniques.

Lemma 5.2 *For any polynomial $p \in \mathcal{P}_r$ also $S_{a_r} p \in \mathcal{P}_r$.*

Proof: Since $\Delta^{r+1} S_{a_r} = 2^{-r} \Delta^{r+1}$, we have for any $p \in \mathcal{P}_r$ that $\Delta^{r+1} S_{a_r} p = 2^{-r} \Delta^{r+1} p = 0$ which is equivalent to $S_{a_r} p \in \mathcal{P}_r$. $\square$

Proposition 5.3 *For a given integer $r > 0$, let $\ell_r(x) = \dfrac{1}{r!} \displaystyle\prod_{j=1}^{r} (x + j)$, then*

$$S_{a_r} \ell_r^{(i)} = \frac{1}{2^{r-i}} \ell_r^{(i)}, \; i = 0, \ldots, r. \tag{38}$$

Proof: *Case $i = 0$:* Let

$$q_r(\alpha) := S_{a_r} \ell_r(\alpha) = \sum_{\beta \in \mathbb{Z}} a_r(\alpha - 2\beta) \ell_r(\beta) \tag{39}$$

By Lemma 5.2, ℓ_r and q_r are two polynomials of degree at most r. Thus it is sufficient to verify the identity $\ell_r = 2^r q_r$ at $r + 1$ points, namely $\alpha = 0, -1, \ldots, -r$.

To begin, we notice that $a_r(\alpha - 2\beta) = 0$ as soon as $\alpha - 2\beta < 0$ or $\alpha - 2\beta > r + 1$. For $\alpha \in \{-1, -2, \ldots, -r\}$, this gives $a_r(\alpha - 2\beta) = 0$ as soon as $\beta \geq 0$ or $\beta < -r$. On the other hand, for $\beta \in [-r, 0) \cap \mathbb{Z}$, we have $\ell_r(\beta) = 0$ so that all the terms in the sum q_r from (39) are vanishing and therefore, $\ell_r(\alpha) = 2^r q_r(\alpha) = 0$, $-r \leq \alpha < 0$. Finally, for $\alpha = 0$, an analogous support argument shows that almost all terms of the sum are vanishing except one, yielding $q_r(0) = a_r(0) \ell_r(0) = 2^{-r} \ell_r(0)$.

To sum up, $2^{-r}\ell_r(\alpha) = q_r(\alpha)$ for $\alpha = 0, -1, \ldots, -r$ which concludes the case.
Case $i > 0$: Since $\frac{1}{2^r}\ell_r = S_{a_r}\ell_r$, we deduce by Lemma 5.1 that

$$\frac{1}{2^r}\ell_r(x) = \frac{1}{2}\sum_{\beta\in\mathbb{Z}} a_r(\beta)\ell_r\left(\frac{x-\beta}{2}\right), \qquad x \in \mathbb{R}.$$

Differentiating i times, this gives

$$\frac{1}{2^r}\ell_r^{(i)}(x) = \frac{1}{2}\frac{1}{2^i}\sum_{\beta\in\mathbb{Z}} a_r(\beta)\ell_r^{(i)}\left(\frac{x-\beta}{2}\right), \qquad x \in \mathbb{R}. \tag{40}$$

Since $S_{a_r}\ell_r^{(i)}$ is a polynomial, Lemma 5.1 and (40) yield

$$S_{a_r}\ell_r^{(i)}(\alpha) = \frac{1}{2}\sum_{\beta\in\mathbb{Z}} a_r(\beta)\ell_r^{(i)}\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2^{r-i}}\ell_r^{(i)}(\alpha), \qquad \alpha \in \mathbb{Z},$$

which completes the proof of (38). $\square$

5.3 Extension to Hermite subdivision schemes

We define Hermite subdivision schemes of degree $d \leq r$ with mask $\{\boldsymbol{A}(\alpha)\}$ and support $[0, r+d+1]$ which are extensions of S_{a_r} by applying differences to the mask a_r, yielding

$$\boldsymbol{A}(\alpha) = \begin{bmatrix} a_r(\alpha) & 0 & \ldots & 0 \\ \Delta a_r(\alpha-1) & 0 & \ldots & 0 \\ \Delta^2 a_r(\alpha-2) & 0 & \ldots & 0 \\ \vdots & & & \\ \Delta^d a_r(\alpha-d) & 0 & \ldots & 0 \end{bmatrix}, \quad \boldsymbol{\mathcal{A}}^*(z) = \frac{(1+z)^{r+1}}{2^r}\begin{bmatrix} 1 & 0 & \ldots & 0 \\ (1-z) & 0 & \ldots & 0 \\ (1-z)^2 & 0 & \ldots & 0 \\ \vdots & & & \\ (1-z)^d & 0 & \ldots & 0 \end{bmatrix}.$$

We begin with $\boldsymbol{f}_0 \in \ell^{d+1}(\mathbb{Z})$ and $\boldsymbol{f}_n \in \ell^{d+1}(\mathbb{Z})$ defined by (6) and notice that
$2^{-(n+1)}f_{n+1}^{(1)}(\alpha) = \sum_{\beta\in\mathbb{Z}} \Delta a_r(\alpha-1-2\beta)f_n^{(0)}(\beta) = \Delta f_{n+1}^{(0)}(\alpha-1)$ so that for $n \geq 1$,

$$f_n^{(1)}(\alpha) = 2^n \Delta f_n^{(0)}(\alpha-1). \tag{41}$$

Similarly for $i = 2, \ldots, d$:

$$f_n^{(i)}(\alpha) = 2^{in}\Delta^i f_n^{(0)}(\alpha-i). \tag{42}$$

Now with (34) and (36), for $n > 0$,

$$\frac{d^i v}{dx^i}(x) = \sum_{\alpha \in \mathbb{Z}} f_n^{(i)}(\alpha)\varphi_{r-i}(2^n x - \alpha), \qquad i = 0, \dots, d.$$

Generally, this Hermite scheme does not satisfy spectral condition, see [19] for an example with $r = 3$ and $d = 2$ but it is possible to get a modified scheme.

Proposition 5.4 *For given integers $r > 0$ and $d \leq r$, there exists an upper triangular matrix $\boldsymbol{R}_{r,d} \in \mathbb{R}^{(d+1)\times(d+1)}$ with 1 on the diagonal and 0 on the first row except the first term such that the spectral condition is satisfied for scheme $S_{\bar{\boldsymbol{A}}}$ with mask $\bar{\boldsymbol{A}}(\alpha) := \boldsymbol{R}_{r,d}^{-1}\boldsymbol{A}(\alpha)\boldsymbol{R}_{r,d}$ with the polynomials $p_{r-i} := \ell_r^{(i)}$ and corresponding eigenvalue $1/2^{r-i}$, $i = r-d, \dots, r$.*

Proof: Since $S_{a_r} p_j = 2^{-j} p_j$ and $v_{p_j}^{(0)} = p_j$, we deduce that

$$\sum_{\beta \in \mathbb{Z}} \boldsymbol{A}(\alpha - 2\beta)\boldsymbol{R}_{r,d}\boldsymbol{v}_{p_j}(\beta) = \begin{bmatrix} \sum_\beta a_r(\alpha - 2\beta) v_{p_j}^{(0)}(\beta) \\ \sum_\beta \Delta a_r(\alpha - 1 - 2\beta) v_{p_j}^{(0)}(\beta) \\ \vdots \\ \sum_\beta \Delta^d a_r(\alpha - d - 2\beta) v_{p_j}^{(0)}(\beta) \end{bmatrix} = \frac{1}{2^j}\begin{bmatrix} p_j(\alpha) \\ \Delta p_j(\alpha - 1) \\ \vdots \\ \Delta^p p_j(\alpha - d) \end{bmatrix}$$

Then

$$S_{\bar{\boldsymbol{A}}} v_{p_j} = \frac{1}{2^j} v_{p_j} \quad \Leftrightarrow \quad \sum_{\beta \in \mathbb{Z}} \boldsymbol{A}(\alpha - 2\beta)\boldsymbol{R}_{r,d}\boldsymbol{v}_{p_j}(\beta) = \frac{1}{2^j}\boldsymbol{R}_{r,d} v_{p_j}$$

$$\Leftrightarrow \quad \begin{bmatrix} p_j(\alpha) \\ \Delta p_j(\alpha - 1) \\ \vdots \\ \Delta^p p_j(\alpha - d) \end{bmatrix} = \boldsymbol{R}_{r,d} \begin{bmatrix} p_j(\alpha) \\ p_j'(\alpha) \\ \vdots \\ p_j^{(d)}(\alpha) \end{bmatrix}.$$

To obtain the components r_{ik} of $\boldsymbol{R}_{r,d}$, we remark that for any polynomial $p \in \mathcal{P}_d$, by the computation of Taylor expansions at point α, $\Delta^i p(\alpha - i) = p^{(i)}(\alpha) + \sum_{k=i+1}^d r_{ik} p^{(k)}(\alpha)$. $\square$

Lemma 5.5 *For any $p \in \mathcal{P}_d$ and $i \leq d$, we have*

$$\Delta^i p(\alpha - i) = p^{(i)}(\alpha) + \sum_{k=i+1}^{d} p^{(k)}(\alpha) \sum_{j=1}^{i} \frac{(-1)^i}{i!}\binom{i}{j}(-1)^j. \tag{43}$$

Proof: We fix α and set

$$\tilde{p} = p - \sum_{j=0}^{i-1} \frac{p^{(j)}(\alpha)}{j!}(\cdot - \alpha)^j$$

so that $p - \tilde{p} \in \mathcal{P}_{i-1}$ and $\tilde{p}^{(k)}(\alpha) = 0$, $k = 0, \ldots, i-1$ as well as $\tilde{p}^{(k)}(\alpha) = p^{(k)}(\alpha)$, $k = i, \ldots, d$. Then, by a Taylor expansion,

$$\begin{aligned}
\Delta^i p(\alpha - i) &= \Delta^i \tilde{p}(\alpha - i) = \sum_{j=0}^{i} \binom{i}{j} (-1)^j \sum_{k=0}^{d} \frac{\tilde{p}^{(k)}(\alpha)}{k!}(-j)^k \\
&= \sum_{j=0}^{i} \binom{i}{j} (-1)^j \sum_{k=i}^{d} \frac{\tilde{p}^{(k)}(\alpha)}{k!}(-j)^k \\
&= \sum_{k=i}^{d} \frac{(-1)^k}{k!} p^{(k)}(\alpha) \sum_{j=0}^{i} \binom{i}{j} (-1)^j j^k =: \sum_{k=i}^{d} r_{ik}\, p^{(k)}(\alpha),
\end{aligned}$$

where the coefficients r_{jk} are independent of p. If we specifically set $p = \ell_r = \frac{1}{r!}x^r + \cdots$ and note that

$$\Delta \ell_r = \frac{1}{r!} \prod_{j=2}^{r} (\cdot + j)\left((x + r + 1) - (x + 1)\right) = \ell_{r-1}(\cdot + 1),$$

it follows that

$$\ell_{r-i} = \sum_{k=1}^{d} r_{ik}\, \ell_r^{(k)}$$

and a comparison of coefficients with respect to x^{r-i} yields $r_{ii} = 1$ and thus (43). $\square$

A nice byproduct of Lemma 5.5 is the combinatorial identity

$$\sum_{j=1}^{k} \binom{k}{j} (-1)^j j^k = (-1)^k k!.$$

In the following subsubsections, we give examples of modified schemes that satisfy spectral conditions and are also extensions of the initial subdivision scheme.

5.4 Cardinal Spline with $r = 4$, $d = 2$

From the Hermite scheme with mask $\boldsymbol{A}$ whose coefficients in the first column are given according to the following table:

α	0	1	2	3	4	5	6	7
$16a_{00}(\alpha)$	1	5	10	10	5	1	0	0
$16a_{10}(\alpha)$	1	4	5	0	-5	-4	-1	0
$16a_{20}(\alpha)$	1	3	1	-5	-5	1	3	1

and $\boldsymbol{R}_{4,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$,

we define the Hermite scheme with mask $\bar{\boldsymbol{A}}(\alpha) = \boldsymbol{R}_{4,2}^{-1}\boldsymbol{A}(\alpha)\boldsymbol{R}_{4,2}$.

With $p_0(x) = 1$, $p_1(x) = x + \frac{5}{2}$, $p_2(x) = \frac{1}{2}x^2 + \frac{5}{2}x + \frac{35}{12}$ and we obtain that $S_{\boldsymbol{A}}\boldsymbol{R}_{4,2}\boldsymbol{b}_j = 2^{-j}\boldsymbol{R}_{4,2}\boldsymbol{b}_j$ or $S_{\bar{\boldsymbol{A}}}\boldsymbol{b}_j = 2^{-j}\boldsymbol{b}_j$ where $\boldsymbol{b}_j = \begin{bmatrix} p_j \\ p_j' \\ p_j'' \end{bmatrix}$, for $j = 0, 1, 2$

5.5 Cardinal spline with $r = 4$, $d = 3$

Now, we build another extension, beginning with mask $\boldsymbol{A}_+(\alpha) \in \mathbb{R}^{4\times 4}$ with

α	0	1	2	3	4	5	6	7	8
$16a_{00}(\alpha)$	1	5	10	10	5	1	0	0	0
$16a_{10}(\alpha)$	1	4	5	0	-5	-4	-1	0	0
$16a_{20}(\alpha)$	1	3	1	-5	-5	1	3	1	0
$16a_{30}(\alpha)$	1	2	-2	-6	0	6	2	-2	-1

and

$$\boldsymbol{R}_{4,3} = \boldsymbol{R}_+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\bar{\boldsymbol{A}}_+(\alpha) = \boldsymbol{R}_+^{-1}\boldsymbol{A}_+(\alpha)\boldsymbol{R}_+$.

With $p_0(x) = 1$, $p_1(x) = x + \frac{5}{2}$, $p_2(x) = \frac{1}{2}x^2 + \frac{5}{2}x + \frac{35}{12}$, $p_3(x) = \frac{1}{6}x^3 + \frac{5}{4}x^2 + \frac{35}{12}x + \frac{25}{12}$, we obtain that $S_{\boldsymbol{A}_+}\boldsymbol{R}_+\boldsymbol{b}_j = 2^{-j}\boldsymbol{R}_+\boldsymbol{b}_j$ or $S_{\bar{\boldsymbol{A}}_+}\boldsymbol{b}_j = 2^{-j}\boldsymbol{b}_j$ where $\boldsymbol{b}_j = \begin{bmatrix} p_j \\ p_j' \\ p_j'' \\ p_j^{(3)} \end{bmatrix}$, for $j = 0, 1, 2, 3$

For $\bar{\boldsymbol{A}}_+$, some straightforward computations yield the *symbol representation*,

$$\bar{\boldsymbol{A}}_+^*(z) = \frac{(1+z)^5}{96}\begin{bmatrix} 6 & 0 & 0 & 0 \\ (z-1)(11-7z+2z^2) & 0 & 0 & 0 \\ 6(z-1)^2(z-2) & 0 & 0 & 0 \\ 6(z-1)^3 & 0 & 0 & 0 \end{bmatrix}$$

and we obtain that $\widetilde{\boldsymbol{B}}_+$ defined by $\widetilde{\boldsymbol{T}}_3\bar{\boldsymbol{A}}_+ = 2^{-3}\widetilde{\boldsymbol{B}}_+\widetilde{\boldsymbol{T}}_3$ has coefficients $\widetilde{\boldsymbol{B}}_+(0) = \mathbf{0}$ and

$$\widetilde{\boldsymbol{B}}_+(7) = \tfrac{1}{72}\begin{bmatrix} -36 & 36 & -18 & 6 \\ -66 & 66 & -33 & 11 \\ -72 & 72 & -36 & 12 \\ -36 & 36 & -18 & 6 \end{bmatrix} = \widetilde{\boldsymbol{B}}_+(8),\ \widetilde{\boldsymbol{B}}_+(5) = \tfrac{1}{72}\begin{bmatrix} 108 & -72 & 0 & 24 \\ 198 & -132 & 0 & 44 \\ 216 & -144 & 0 & 48 \\ 108 & -72 & 0 & 24 \end{bmatrix} = \widetilde{\boldsymbol{B}}_+(6),$$

$$\widetilde{\boldsymbol{B}}_+(3) = \tfrac{1}{72}\begin{bmatrix} -108 & 36 & 18 & 6 \\ -198 & 66 & 33 & 11 \\ -216 & 72 & 36 & 12 \\ -108 & 36 & 18 & 6 \end{bmatrix} = \widetilde{\boldsymbol{B}}_+(4),\ \widetilde{\boldsymbol{B}}_+(1) = \frac{1}{72}\begin{bmatrix} 36 & 0 & 0 & 0 \\ 66 & 0 & 0 & 0 \\ 72 & 0 & 0 & 0 \\ 36 & 0 & 0 & 0 \end{bmatrix} = \widetilde{\boldsymbol{B}}_+(2).$$

We use the same tools as in the previous section, building the two matrices in $\mathbb{R}^{32\times 32}$:

$$\boldsymbol{U}_0 := \begin{bmatrix} \widetilde{\boldsymbol{B}}_+(1) & \widetilde{\boldsymbol{B}}_+(-1) & \mathbf{0} & \dots & \mathbf{0} \\ \widetilde{\boldsymbol{B}}_+(2) & \widetilde{\boldsymbol{B}}_+(0) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & & & \vdots \\ \widetilde{\boldsymbol{B}}_+(7) & \widetilde{\boldsymbol{B}}_+(5) & \widetilde{\boldsymbol{B}}_+(3) & \dots & \mathbf{0} \\ \widetilde{\boldsymbol{B}}_+(8) & \widetilde{\boldsymbol{B}}_+(6) & \widetilde{\boldsymbol{B}}_+(4) & \dots & \mathbf{0} \end{bmatrix},$$

$$\boldsymbol{U}_1 := \begin{bmatrix} \widetilde{\boldsymbol{B}}_+(2) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \widetilde{\boldsymbol{B}}_+(3) & \widetilde{\boldsymbol{B}}_+(1) & \mathbf{0} & & \\ \vdots & \vdots & & & \vdots \\ \widetilde{\boldsymbol{B}}_+(8) & \widetilde{\boldsymbol{B}}_+(6) & \widetilde{\boldsymbol{B}}_+(4) & \dots & \mathbf{0} \\ \mathbf{0} & \widetilde{\boldsymbol{B}}_+(7) & \widetilde{\boldsymbol{B}}_+(5) & \dots & \mathbf{0} \end{bmatrix}$$

Again for a positive integer p, if $M_{\boldsymbol{u}} := \Pi_{k=1}^{p}\boldsymbol{U}_0^{u_k}\boldsymbol{U}_1^{1-u_k}$, for all $u_k \in \{0,1\}$ and $\boldsymbol{u} =: [u_k]_{k=1,\dots,p}$, if $\rho_p := \max_{\boldsymbol{u}\in\{0,1\}^p}\|M_{\boldsymbol{u}}\|_1^{1/p}$, then a numerical computation gives $\rho_8 = 0.9173$. Therefore, the operator $S_{\widetilde{\boldsymbol{B}}_+}$ is contractive, proving that the Hermite scheme $S_{\bar{\boldsymbol{A}}_+}$ is C^3. Now, since $\boldsymbol{R}_{4,3}$ is triangular with 1 on its diagonal, it is easy to deduce that for $r=4$ and $d=3$, the scheme $S_{\boldsymbol{A}}$ and S_{a_4} are C^3 which recovers a well known property of cardinal splines.

References

[1] A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, *Stationary Subdivision*, Mem. Amer. Math. Soc. **453** (1991).

[2] D. R. Chen, Construction of smooth refinable vectors by cascade algorithms , Num. Anal. **40** (2002), pp.1354–1368.

[3] C. Conti and K. Jetter, *Concerning Order of Convergence for Subdivision*, Num. Algo, **36**, (2004), 345–363.

[4] M. Charina, C. Conti and T. Sauer, *Regularity of Multivariate Vector Subdivision Schemes*, Num. Algo, **39**, (2005), 97–113.

[5] C. Conti, K. Hormann, *Polynomial reproduction for univariate subdivision schemes of any arity*, *J. Approx. Theory*, **163** (2011), 413–437.

[6] C. Conti, J.-L.. Merrien, L. Romani, *Dual Hermite subdivision schemes of de Rham-type*, BIT Num Math, **54** (2014), 955-977

[7] S. Dubuc and J.-L. Merrien, *de Rham Transform of a Hermite Subdivision Scheme*, Approximation Theory XII, San Antonio 2007, M. Neamtu and L.L. Schumaker eds, 2008, Nashboro Press, Nashville TN, 121-132.

[8] S. Dubuc and J.-L. Merrien, *Hermite subdivision schemes and Taylor polynomials*, Constr. Approx. **29** (2009), 219–245.

[9] N. Dyn, D. Levin, *Analysis of Hermite-type subdivision schemes*, (C. K. Chui and L.L. Schumaker Eds.), Approximation Theory VIII. Vol 2: Wavelets and Multilevel Approximation (College Station, TX, 1995). World Sci., River Edge, NJ, 1995, pp. 117-124.

[10] N. Dyn, D. Levin, *Analysis of Hermite-interpolatory subdivision schemes*, (S. Dubuc et G. Deslauriers, Eds.), Spline Functions and the Theory of Wavelets, Amer. Math. Soc., Providence R. I., 1999, pp. 105-113.

[11] N. Dyn and D. Levin, *Subdivision schemes in geometric modelling*, Acta Numerica **11** (2002), 73–144.

[12] N. Dyn, K. Hormann, M. A. Sabin, and Z. Shen, *Polynomial reproduction by symmetric subdivision schemes*, *J. Approx. Theory* **155** (2008), 28–42.

[13] N. Guglielmi, C. Manni, D. Vitale, *Convergence analysis of C^2 Hermite interpolatory subdivision schemes by explicit joint spectral radius formulas*, Linear Algebra and its Applications **434** (2011) 884-902

[14] B. Han, *Vector cascade algorithms and refinable functions in sobolev spaces*, J. Approx. Theory **124** (2003), 44–88.

[15] B. Han, T. Yu, and Y. Xue, *Noninterpolatory Hermite subdivision schemes*, Math. Comp. **74** (2005), 1345–1367.

[16] J.-L. Merrien, *A family of Hermite interpolants by bisection algorithms*, Numer. Algo. **2** (1992), 187–200.

[17] J.-L. Merrien, *Interpolants d'Hermite C2 obtenus par subdivision*, M2AN Math. Model. Numer. Anal. **33** (1999) 55-65.

[18] J.-L. Merrien, T. Sauer, *From Hermite to stationary subdivision schemes in one or several variables*, Adv. Comput. Math. **36** (2012) 547-579

[19] J.-L. Merrien, T. Sauer, *A Generalized Taylor Factorization for Hermite Subdivisions Schemes*, J. of Comp. & App. Math. **236** (2011), 565-574.

[20] C. A. Micchelli, *Mathematical Aspects of Geometric Modelling.* SIAM, Philadelphia, 1995.

[21] C.A. Micchelli, T. Sauer, *On vector subdivision*, Math. Z. **229** (1998), 621–674.

[22] I. J. Schoenberg, *Cardinal spline interpolation*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 12, SIAM, 1973.